Combinatorics Worksheet 4: Permutations

1. (a) 20 people audition for a play with 10 roles. How many ways are there to choose a cast for the play?

Solution: The only thing to be careful about here is that the roles are all distinct. So this is just the number of 10-permutations of 20, or $\boxed{20 \cdot 19 \cdot 18 \cdot 17 \cdots 11 = \frac{20!}{10!}}$.

(b) What if actors are allowed to play more than one role each?

Solution: Since we can now reuse actors, there are 20 options for every role. Since we have to make this choice 10 times, there are $\boxed{20^{10}}$ ways to cast the play.

2. How many ways are there to arrange a deck of 52 cards so that for each suit, all cards of that suit are together (there are 4 suits of 13 cards each)?

Solution: There are a few ways to approach this problem.

One method is to imagine first choosing which order to put the 4 suits in and then, for each suit, choosing how to arrange the 13 cards in the suit. There are 4! ways the 4 suits could be put arranged and for each suit there are 13! ways to arrange the cards in that suit (and we have to repeat this 4 times, once for each suit). So there are $\boxed{4!(13!)^4}$ ways to arrange the cards.

Common Mistakes: A common mistake when using method one is to forget that the cards of *each* suit have to be put in order and end up with the **incorrect** answer of $4! \cdot 13!$.

Solution: Another method is to imagine arranging all 52 cards, one at a time. For the first position, we can choose any card in the deck so we have 52 options. The next card needs to be the same suit as the first one, but it can be any of the remaining cards in that suit, so we have 12 options. This continues until we've finished the first suit. Then we can choose any of the remaining $52 - 13 = 39$ cards. But for the position after that we once again need to be in the same suit as the previous card, and so on. Tallying it all up, we end up with

$$52 \cdot 12 \cdot 11 \cdots 2 \cdot 1 \cdot 39 \cdot 12 \cdot 11 \cdots 2 \cdot 1 \cdot 26 \cdot 12 \cdots 1 \cdot 13 \cdot 12 \cdots 1 = 52 \cdot 39 \cdot 26 \cdot 13 \cdot (12!)^4$$

ways to arrange the deck. It is true, though not obvious at first glance, that this is equal to the solution from the first method.

3. Determine the larger number in each pair below. Feel free to experiment on a calculator.

(a) The number of permutations of a set of size n or the number of subsets of a set of size n (where n is greater than 1)?

Solution: There are more permutations.

(b) The number of 5-permutations of a set of size n or the number of subsets of a set of size n (where n is very large)?

Solution: There are more subsets.

(c) The number of 5-permutations of a set of size n or the number of $(n-5)$-permutations of a set of size n (when $n > 10$)?

Solution: There are more $(n-5)$-permutations.

4. Could you plausibly write down all permutations of a set of 5 elements? What about 10? What about 20? How many years would it take to write permutations of 12 elements?

Solution:

- The number of permutations of a set of size 5 is $5! = 120$ and each item on the list has 5 elements, so this is annoying but very doable.
- The number of permutations of a set of size 10 is $10! = 3628800$ and each item on the list has 10 elements, so it is doubtful that most people would have enough patience to do this, but it is definitely possible for one human to do this in a year if they worked on it every day (some novels, for instance, are over a million words).
- The number of permutations of a set of size 20 is $20! \approx 2.4 \times 10^{18}$ and each item on the list has 20 elements, so this is totally absurd (for instance, this is more than the number of milliseconds that have elapsed since the first human being was born).
- The number of permutations of a set of size 12 is $12! = 479001600$ and each item on the list has 12 elements. I timed myself writing the numbers 1 through 12 and it took me about 6 seconds. But let's be generous and say that each permutation can be written in 5 seconds. So we need about $5 \cdot 12!$ seconds to write all the permutations, which works out to about 76 years. If you include time to sleep, eat, use the bathroom and so on, then this number should at least be doubled. So 12 is probably the lowest number for which a person could not plausibly write all the permutations.

5. Explain why it is not a good idea in Scrabble to simply try out all possible moves each turn.

Solution: A scrabble player has 7 tiles and even ignoring the different positions they can be placed on the board and the fact that not all the tiles have to be used, this already gives $7! = 5040$ possible moves (many of them, of course, are illegal). The actual number of possible moves is many times higher than this, but even if you can try out one move per second, 5040 moves is already far more than enough to try the patience of any Scrabble player.

6. Each square of a 3×7 grid of squares is colored either red or blue. Show that there must be a rectangle in the grid whose corner squares are all the same color. (Hint: apply pigeonhole principle multiple times.)

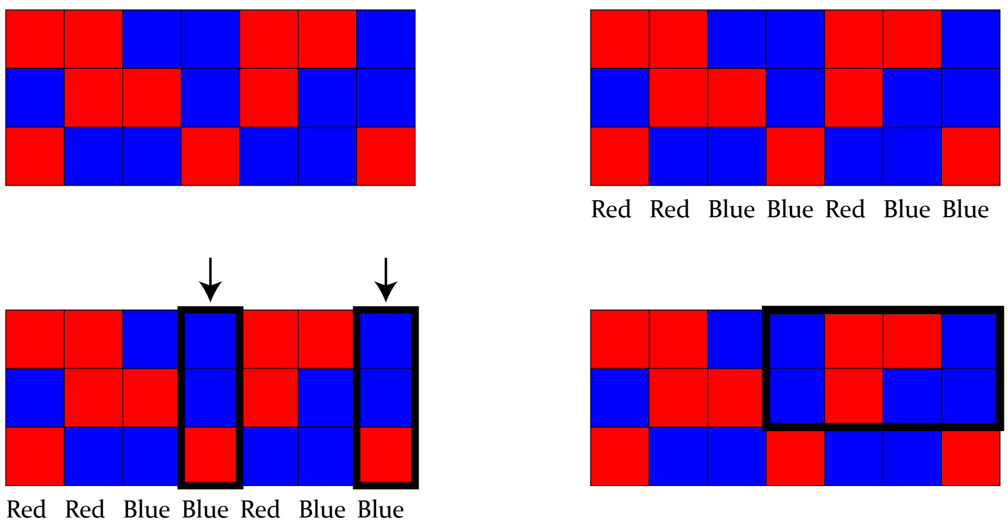

Figure 1: Example for the solution to problem 6

Comment: Some people cleverly observed that if rectangles are allowed to consist of just two squares then the problem is not too difficult (and if we allow rectangles consisting of just one square then the problem becomes totally trivial). So let's assume that rectangles must include at least 4 squares.

Solution: First observe that by the pigeonhole principle, any given column of the grid must have at least two squares of the same color (since there are three squares per column, but only two squares). Label a column "Red" if it contains at least two red squares and "Blue" otherwise.

Next, the generalized pigeonhole principle tells us that at least 4 of the columns must have the same label, since there are 7 columns and only two labels. For convenience, let's suppose that at least 4 columns are labelled "Red" (though the argument is the same if they are instead labelled "Blue").

We know that each of the four "Red" columns has at least two red squares. If any column has all three squares red, then we can pick any other column with at least two red squares and use these, along with the corresponding two squares in the all-red column, to form the corners of a rectangle. Suppose instead that every "Red" column has exactly two red squares. There are only 3 ways to color two out of three squares red, so by the pigeonhole principle, there are two "Red" columns whose red squares are in matching positions. And we can use these four red squares to form the corners of a rectangle.

Comment: An example of this process and the resulting rectangle is shown in figure 1 for one particular way the squares of the grid could be colored. Note that an example like this can be very helpful but can not be *substituted* for an explanation like the one above.

7. **Challenge Problem:** Show that for any set of 100 integers, there is some non-empty subset whose sum is a multiple of 100.