Combinatorics Worksheet 2: Inclusion-Exclusion

1. How many numbers below 100 are divisible by 2,3, or 5?

- Let A be the set of positive integers less than 100 that are divisible by 2.
- Let B be the set of positive integers less than 100 that are divisible by 3.
- Let C be the set of positive integers less than 100 that are divisible by 5.

We want to find the size of $A \cup B \cup C$. The size of each is easy to find on its own, but the size of the union is kind of tricky to calculate directly, so we will use the technique of inclusion-exclusion.

- $|A| = \lfloor \frac{99}{2} \rfloor = 49$
- $|B| = \lfloor \frac{99}{3} \rfloor = 33$
- $|C| = \lfloor \frac{99}{5} \rfloor = 19$
- The set $A \cap B$ is the set of positive integers less than 100 that are divisible by both 2 and 3, which is equivalent to being divisible by 6. So $|A \cap B| = \lfloor \frac{99}{6} \rfloor = 16$
- $|A \cap C| = \lfloor \frac{99}{10} \rfloor = 9$
- $|B \cap C| = \lfloor \frac{99}{15} \rfloor = 6$
- $|A \cap B \cap C| = \lfloor \frac{99}{30} \rfloor = 3$

The principle of inclusion-exclusion now tells us that $|A \cup B \cup C| = \boxed{49 + 33 + 19 - 16 - 9 - 6 + 3 = 73}$.

2. (a) Out of a class of 20 students, how many ways are there to form a study group? Assume that a study group must have at least 2 students.

This is equal to the number of subsets of a set of size 20 which have size at least 2. There are 2^{20} total subsets, 20 of which have size 1 and 1 of which has size 0. So the answer is $\boxed{2^{20} - 20 - 1}$.

(b) How many ways are there to form a study group that contains at least one of Bob, Sue, and Alicia?

There are several ways to approach this problem. One uses inclusion-exclusion, but there is also another, slightly simpler, solution.
First, the inclusion-exclusion solution.

- Let A be the set of study groups that contain Alicia
- Let B be the set of study groups that contain Bob
- Let C be the set of study groups that contain Sue

We want to find the size of $A \cup B \cup S$.

- To form a study group that contains Alicia, we just need to choose which of the 19 other students to also include. There are 2^19 ways to do this, but one of the ways consists of choosing nobody. This would result in a study group with just Alicia and since study groups must have at least 2 people, we will leave it out. This gives $|A| = 2^{19} - 1$. By the same reasoning, $|B| = 2^{19} - 1$ and $|S| = 2^{19} - 1$.

- To form a study group that contains both Alicia and Bob we just need to choose which of the 18 other students to also include. There are 2^{18} ways to do this and we don't need to leave any out because Alicia and Bob together already form a group of size 2. So $|A \cap B| = 2^{18}$ and by the same reasoning $|A \cap S| = 2^{18}$ and $|B \cap S| = 2^{18}$.

- Similar reasoning also shows that $|A \cap B \cap S| = 2^{17}$.

By the principle of inclusion-exclusion, $|A \cup B \cup S| = \boxed{3 \cdot (2^{19} - 1) - 3 \cdot 2^{18} + 2^{17}}$.

Now for the other solution. Instead of counting study groups that include at least one of Alicia, Bob, and Sue, we will count study groups that don't include any of Alicia, Bob, or Sue. To form such a study group, we just need to choose at least 2 of the remaining 17 students. By the same reasoning we used in part (a), there are $2^{17} - 17 - 1$ ways to do this. To find the answer to the original question, we need to subtract this from the total number of study groups. This gives us $\boxed{2^{20} - 21 - (2^{17} - 18) = 2^{20} - 2^{17} - 3}$. It is straightforward to check that this is the same answer given by the first method.

3. Suppose you need to come up with a password that uses only the letters A, B, and C and which must use each letter at least once. How many such passwords of length 8 are there?

We will first find the number of passwords that leave out at least one of A, B, or C.

- Let X be the set of passwords that don't contain A
- Let Y be the set of passwords that don't contain B
- Let Z be the set of passwords that don't contain C

We want to find the size of $X \cup Y \cup Z$.

- Passwords that don't contain A just contain B and C. So there are 2^8 such passwords—i.e. $|X| = 2^8$. By the same reasoning $|Y| = |Z| = 2^8$.

- Passwords that don't contain A or B just contain C. There is one such password (namely 'CCCCCCCC') so $|X \cap Y| = 1$. By the same reasoning $|X \cap Z| = |Y \cap Z| = 1$.

- Passwords that don't contain A, B, or C can't exist because passwords in this problem only use the letters A, B, and C. So $|X \cap Y \cap Z| = 0$.

So by inclusion-exclusion, $|X \cup Y \cup Z| = 3 \cdot 2^8 - 3 \cdot 1 + 0 = 3 \cdot 2^8 - 3$.
To find the answer to the original question, we need to subtract the number we just found from the total number of passwords, which is 3^8. This gives $\boxed{3^8 - (3 \cdot 2^8 - 3)}$.

4. (a) Suppose that 4 people are standing in line. How many ways are there to rearrange the line so that nobody is standing in their original place?

We will first count the number of ways to arrange the line of people so that at least one person stays in the same place.
For each i between 1 and 4, let A_i be the set of ways to order the people in line so that person i stays in the same place. We want to find the size of $A_1 \cup A_2 \cup A_3 \cup A_4$.
The number of ways to order the line so that person i remains in the same place is just the number of ways to arrange the other three people—i.e. $3 \cdot 2 \cdot 1 = 3!$. In other words, for each i, $|A_i| = 3!$. Similar reasoning shows that the double intersections have size $2!$, and that the triple and quadruple intersections have size 1.
Since there are 6 double intersections, 4 triple intersections, and 1 quadruple intersection, the inclusion-exclusion tells us that the number of ways to arrange the people so that someone stays in the same place is $4 \cdot 3! - 6 \cdot 2! + 4 \cdot 1 - 1 \cdot 1$.
Subtracting this from the total number of ways to arrange four people, which is $4!$, gives us $\boxed{4! - (4 \cdot 3! - 6 \cdot 2! + 4 \cdot 1 - 1 \cdot 1) = 9}$.

(b) **Challenge Problem:** Suppose n people are standing in line and are then randomly put into a new order. When n is very large, approximately what is the probability that nobody is standing in their original place? (Occasionally I will put challenge problems on the worksheets. I will not go over them in class or post solutions for them, but I encourage you to try to solve them and to ask me about them in office hours.)

If you think you have a solution or if you just want to talk about the problem with me, come to office hours.

5. (a) How many functions from $\{1, 2, 3, 4, 5\}$ to $\{A, B, C\}$ are there?

A function from $\{1, 2, 3, 4, 5\}$ to $\{A, B, C\}$ is just a way of assigning one of A, B, C to each of 1, 2, 3, 4, 5. So for each element of $\{1, 2, 3, 4, 5\}$ we need to make a choice with 3 possible options. And there are $\boxed{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^5}$ ways of making all 5 choices.

(b) How many functions from $\{1, 2, 3, 4, 5\}$ to $\{A, B, C\}$ are onto?

This is actually almost exactly the same question as problem number 3, except that here 8 is replaced by 5. So the solution is $\boxed{3^5 - (3 \cdot 2^5 - 3)}$.