# Final Review Worksheet 1 Solutions

## Final Topics

Below is a list of the main topics and types of problems we saw in this class. Bold means that I think that item is fairly likely to show up on the final exam. I cannot promise that this list is exhaustive. Also keep in mind that I have not seen the final exam and so I cannot be certain what questions will appear on it.

- **Combinatorics problems involving addition, multiplication, permutations, combinations**
- **Boxes and balls: stars and bars, stars and bars when every box has to have a ball, Stirling numbers**
- Pigeonhole principle
- Inclusion-Exclusion
- Discrete probability problems that use combinatorics
- **Definition of conditional probability**
- **Bayes' rule for calculating conditional probability**
- Definition of independent events and independent random variables
- Definition of expected value and variance
- Linearity of expectation
- **Distributions (probability mass function, expected value, variance): Bernoulli, binomial, geometric, hypergeometric, Poisson**
- Central limit theorem and law of large numbers
- **Finding 95% confidence intervals, including estimating mean and standard deviation for random variables following Bernoulli, binomial, Poisson, etc. distributions**
- **Hypothesis testing: $\chi^2$ tests for goodness of fit and $\chi^2$ tests for independence. How to perform them, how to interpret the results.**
- **How to solve recurrence relations, with and without initial conditions**
- **How to solve ordinary differential equations: separation of variables, integrating factor, and characteristic polynomial**
- Linear differential equations: definition, how to check if something is linear, and why it is a useful property.
- **Partial fractions for finding some integrals**
- Given some scenario (often involving a tank of salt water for some reason), write a differential equation to model the situation.
- **Euler's method**
- Slope fields
- How to add and multiply matrices, transpose of a matrix, multiplying matrices by scalars and vectors.
- Finding the determinant of a matrix
- **Solving systems of linear equations using Gaussian elimination**
- **Finding the inverse of a matrix using Gaussian elimination**
- **Finding eigenvalues and eigenvectors of a matrix**
- **Finding solutions to systems of linear ODEs (with or without initial conditions) using eigenvalues and eigenvectors**
- Least squares method for linear regression

## Review Problems (Combinatorics and Probability)

1. Suppose the final exam is worth 21 points and has six questions.

   (a) How many ways are there to allocate the points? Every question must be worth a positive integer number of points.

   **Solution:** Think of the points as indistinguishable balls and the questions as distinguishable boxes. The only requirement is that every box must have at least one ball (since no question can be worth 0 points). If we first throw one ball into each box, we are left with $21 - 6 = 15$ balls to distribute in any way among the 6 boxes. To find the number of ways to do this, we can use stars and bars, giving us $\boxed{\binom{15+6-1}{15}}$.

   (b) How many ways are there to allocate the points if, in addition to the restrictions in part (a), no question can be worth more than half the total number of points?

   **Solution:** It is easiest to first count the number of ways to assign points where some question *does* get more than half the total number of points, and then subtract this from the answer to part (a).

   First note that it is impossible for two questions to be assigned more than half the number of points. So to assign points so that one question gets more than half the total number of points—i.e. at least 11 points—we can proceed as follows: first pick which question will get at least 11 points. Then assign 11 points to that question and one point to each other question. Finally, distribute the remaining $21 - 11 - 5 = 5$ points among the 6 questions in any way.

   There are 6 ways to complete the first step, 1 way to complete the second step and to calculate the number of ways to complete this final step we can use stars and bars. This gives us

   $$6\binom{5+6-1}{5}$$

   Subtracting this from our answer to part (a) gives the final answer of

   $$\boxed{\binom{15+6-1}{15} - 6\binom{5+6-1}{5}}.$$

2. Suppose you have 20 dogs and 5 cats. You want to put them in a line, but you must have at least 2 dogs between any two cats. How many ways are there to do this? (The dogs and cats are distinguishable.)

   **Solution:** Let's think of lining up the dogs and cats as a task we can complete. The procedure to complete this task is as follows:

   1. Choose which order the cats will go in.

2. Choose how many dogs will go in each space between two cats (at least two dogs must go in each of these spaces) and in the spaces before and after all the cats.

3. Choose which order the dogs will go in.

There are 5! ways to complete the first step and 20! ways to complete the last step. For the second step, we can use stars and bars. There are 4 spaces between two cats and 2 more spaces at the ends of the line of cats (i.e. before all the cats and after all the cats). We can think of these spaces as distinguishable boxes. And since we are just deciding how many dogs go in each space, we can think of the dogs as indistinguishable balls (and in step three we will actually put them in order). So we can use stars and bars. First we put two balls into each of the 4 middle boxes and then we can distribute the remaining $20 - 4 \cdot 2 = 12$ balls in any way among all 6 boxes. There are

$$\binom{12+6-1}{12}$$

ways to do this, so the final answer is $\boxed{5!20!\binom{12+6-1}{12}}$.

3. Suppose you draw 10 cards from a standard 52-card deck.

(a) What is the probability that you get 10 different ranks?

**Comment:** Originally, this problem read "What is the probability that you get 10 different **suits**?" The answer to that question is 0, because there are only 4 suits in the deck. The intended question, and the one to which a solution is given below, instead asks about ranks.

**Solution:** Let's let the sample space, $\Omega$, be the set of all unordered sets of 10 distinct cards. Let $A$ be the event that all 10 cards are of different ranks. Since each outcome in $\Omega$ is equally likely, we can find $P(A)$ simply by counting the sizes of both $A$ and $\Omega$.

The size of $\Omega$ is just $\binom{52}{10}$.

To calculate the size of $A$, imagine a procedure for forming sets of 10 cards of all different ranks. One such procedure is as follows:

1. Choose 10 ranks out of the 13 possible ranks.

2. For each rank chosen, choose one of the 4 possible suits.

There are $\binom{13}{10}$ ways to complete the first step and $4^{10}$ ways to complete the second step (since we make a choice with 4 possible options 10 times in a row). So $|A| = 4^{10}\binom{13}{10}$ and thus

$$P(A) = \frac{4^{10}\binom{13}{10}}{\binom{52}{10}}$$

(b) What is the probability that you get 6 cards of one suit, and 2 cards each of two other suits?

**Solution:** As in part (a), we just need to count the number of hands satisfying this description. Let's first imagine a procedure for constructing such a hand:

1. Choose one suit out of the 4 possible suits to be the suit with 6 cards
2. Choose 2 of the remaining 3 suits to be the suits with 2 cards each
3. Choose 6 out of the 13 possible ranks for the first suit
4. For each of the other 2 suits, choose 2 out of the 13 possible ranks

There are 4 ways to complete step 1, $\binom{3}{2}$ ways to complete step 2, $\binom{13}{6}$ ways to complete step 3, and $\binom{13}{2}^2$ ways to complete step 4. Therefore the probability is

$$\frac{4 \cdot \binom{3}{2}\binom{13}{6}\binom{13}{2}^2}{\binom{52}{10}}$$

(c) What is the probability that you get a spade given that you didn't get any hearts?

**Solution:** Let $S$ be the event that you get at least one spade and $H$ be the event that you get at least one heart. Then we are trying to find $P(S \mid H^C)$. By the definition of conditional probability, this is equal to

$$P(S \mid H^C) = \frac{P(S \cap H^C)}{P(H^C)}$$

We will calculate both these terms. First, $P(H^C)$ is just

$$\frac{\binom{39}{10}}{\binom{52}{10}}$$

because there are $\binom{39}{10}$ ways to choose 10 cards that are all not hearts. To find $P(S \cap H^C)$ it is easier to instead find $P(S^C \cap H^C)$ and then use the fact that $P(S \cap H^C) = P(H^C) - P(S^C \cap H^C)$. This gives us

$$P(S \cap H^C) = \frac{\binom{39}{10}}{\binom{52}{10}} - \frac{\binom{26}{10}}{\binom{52}{10}}$$

because there are $\binom{26}{10}$ ways to choose 10 cards that are all not hearts and not spades. So the final answer is

$$\frac{\binom{39}{10} - \binom{26}{10}}{\binom{39}{10}} = 1 - \frac{\binom{26}{10}}{\binom{39}{10}}$$

Another way to solve this problem is to think of it as a question about a reduced sample space which only includes hands with no hearts.

4. Suppose you have 3 fair dice with 3, 4, and 6 sides (and whose sides are numbered 1–3, 1–4, and 1–6, respectively). You randomly pick one of the dice and roll it.

   (a) Let the random variable $X$ be the number that you rolled. What is $E[X]$?

   **Solution:** This can be found using the definition of expected value. The answer is

   $$\frac{8}{3} \approx 2.667$$

   (b) If you roll a number less than 4, what is the probability that you picked the 6-sided die?

   **Solution:** $\boxed{\frac{2}{9}}$