Matrix Algebra Worksheet 1 Solutions

1. (a) Write a 2×4 matrix.

Solution: Here's one possible answer:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

(b) Write a 4×2 matrix.

Solution: Here's one possible answer:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(c) Multiply the two matrices that you just wrote.

Solution: The problem does not state which order to multiply the matrices in (and either order is valid here) so let's just pick an order.

$$\begin{aligned} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 1\cdot 1 + 1\cdot 2 + 0\cdot 3 + 0\cdot 4 & 1\cdot 1 + 0\cdot 2 + 1\cdot 3 + 0\cdot 4 \\ 1\cdot 1 + 1\cdot 0 + 0\cdot 0 + 0\cdot 2 & 1\cdot 1 + 0\cdot 0 + 1\cdot 0 + 0\cdot 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

If we had multiplied the matrices in the other order, we would have instead ended up with a 4×4 matrix.

(d) Write two matrices that you can't multiply.

Solution: There are many possible answers here. We just need two matrices so that the number of columns of the first matrix is not equal to the number of rows in the second. One example is

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 4 \\ 0 & 1 & 5 \end{bmatrix}$$

(e) Write two matrices that you can multiply in one order but not in the other order.

Solution: We need two matrices so that the number of columns of the first one is not equal to the number of rows of the second, but the number of columns of the second *is* equal to the number of rows of the first. One example is

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 4 \\ 0 & 1 & 5 \end{bmatrix}$$

(f) Write a matrix that is equal to its transpose.

Solution: First observe that if a matrix has dimensions $n \times m$ then its transpose has dimensions $m \times n$. So if the matrix is equal to its transpose, we must have $n = m$—i.e. the matrix must be square.

Furthermore, since the rows of the transpose are just the columns of the original matrix, we must find a matrix whose first column is the same as its first row, second column is the same as its second row, and so on. So one way to find such a matrix is as follows: First fill in the first row and then fill in the first column to match it. Then fill in the remainder of the second row and fill in the second column to match it, and so on.

One example of such a matrix is

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

(g) Write a square matrix that is not invertible.

Solution: We need a matrix whose determinant is zero. One such matrix is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Another example is

$$\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$$

(h) Write a square matrix that is invertible.

Solution: We need a matrix whose determinant is not zero. One example is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Another one is

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

(i) Write 2×2 matrices A, B, C such that $AB = AC$ but $B \neq C$.

Solution: First, a helpful observation. Suppose A is invertible and $AB = AC$. Then multiplying both sides by A^{-1} gives $A^{-1}(AB) = A^{-1}(AC)$. Using distributivity and the fact that $A^{-1}A = I$, this implies that $IB = IC$ and therefore $B = C$.

So if we want to solve this problem, we need to pick some A that is *not* invertible. The simplest option is

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

In which case, we can choose B and C to be any two matrices. For instance

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Here's another example.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 10 & 11 \\ 0 & 1 \end{bmatrix}$$

(j) Write 2×2 matrices A and B such that $AB \neq BA$.

Solution: Almost any two matrices you write down will work. For instance, if

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

then

$$AB = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

but

$$BA = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

(k) Write a diagonal matrix.

Solution: Recall that "diagonal" just means everything not on the diagonal is zero. The entries on the diagonal are allowed to be zero or nonzero. A few possible answers:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

(l) Write an upper triangular matrix.

Solution: Recall that "upper traingular" just means everything below the upper triangle is zero. A few possible answers:

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 6 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

(m) Write a system of linear equations with infinitely many solutions.

Solution:

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 2 \end{aligned}$$

(n) Write a system of linear equations with exactly one solution.

Solution:

$$\begin{aligned} x &= 1 \\ y &= 2 \end{aligned}$$

(o) Write a system of linear equations with no solution.

Solution:

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 3 \end{aligned}$$

2. True or false:

(a) The following matrix is diagonal

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution: True. All entries not on the diagonal are zero.

(b) The following matrix is upper triangular

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & -4 \end{bmatrix}$$

Solution: True. All entries not on the upper triangle are zero.

(c) If the determinant of a square matrix is not zero, then the matrix is invertible.

Solution: True. This was mentioned in lecture.

(d) If the coefficient matrix is not invertible then a system of linear equations cannot have a solution.

Solution: False. For instance, consider the following system of linear equations

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 2 \end{aligned}$$

There is clearly a solution (in fact, there are infinitely many solutions) but the coefficient matrix is

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

which is not invertible.

3. Find all solutions of the following system of linear equations.

$$\begin{array}{rcrcrcr} & & 4x_2 & + & 8x_3 & = & 12 \\ x_1 & - & x_2 & + & 3x_3 & = & -1 \\ 3x_1 & - & 2x_2 & + & 5x_3 & = & 6 \end{array}$$

Solution: First, let's write the corresponding augmented matrix.

$$\left[\begin{array}{ccc|c} 0 & 4 & 8 & 12 \\ 1 & -1 & 3 & -1 \\ 3 & -2 & 5 & 6 \end{array}\right]$$

Now we can use Guassian elimination

$$\left[\begin{array}{ccc|c} 0 & 4 & 8 & 12 \\ 1 & -1 & 3 & -1 \\ 3 & -2 & 5 & 6 \end{array}\right] \xrightarrow{\text{Switch } R_1 \text{ and } R_2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 4 & 8 & 12 \\ 3 & -2 & 5 & 6 \end{array}\right] \xrightarrow{R_3 - 3R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 4 & 8 & 12 \\ 0 & 1 & -4 & 9 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & -4 & 9 \end{array}\right] \xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -6 & 6 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{6}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Now we use back-substitution to find the solution. The system of equations corresponding to the last augmented matrix above is:

$$\begin{aligned} x_1 - x_2 + 3x_3 &= -1 \\ x_2 + 2x_3 &= 3 \\ x_3 &= -1 \end{aligned}$$

So we know $x_3 = -1$. Plugging this into the second equation, we see that $x_2 = 3 - 2(-1) = 5$. Plugging both of these into the first equation, we get $x_1 = -1 + x_2 - 3x_3 = -1 + 5 - 3(-1) = 7$. So the final answer is $\boxed{x_1 = 7,\ x_2 = 5,\ x_3 = -1}$.