Combinatorics Worksheet 1

1. How many 4 digit numbers are there which contain only even digits?

   There are 5 even digits (0, 2, 4, 6, 8). But the first digit of a number is never 0. So to form a four digit number with only even digits, we have 4 choices for the first digit and 5 choices for each of the three remaining digits. So there are $4 \cdot 5 \cdot 5 \cdot 5 = \boxed{4 \cdot 5^3}$ such numbers.

2. Suppose you have 10 colors of paint. You can also combine these colors to form new colors. Each combination of the ten original colors gives a different color of paint, but proportions don't matter (so a little green mixed with a lot of red gives the same color as a little red mixed with a lot of green). How many colors can you create, including the original ten colors? (By the way, for the purposes of this problem the absence of paint does not count as a color.)

   For each color of paint, we need to choose whether to include it in the mix or not. So we have to make 10 choices, with two possibilities for each choice. This gives us 2^{10} ways overall to make all the choices. There is one small problem, however: the 2^{10} includes the possibility that we don't include any paint color, which is not a valid choice in this problem. So the total number of colors that we can create is $\boxed{2^{10} - 1}$.

3. (a) How many possible answer keys are there for a 30 question true/false test?

   Each answer can either be true or false. So there are 30 choices with two possibilities each and thus the total number of answer keys is $\boxed{2^{30}}$.

   (b) How many possible ways are there for a student to fill out a 30 question true/false test? (The student does not need to answer every question.)

   This is the same as part (a) except that now there are three possibilites for each question (true, false, or blank). So the answer is $\boxed{3^{30}}$.

4. Suppose you own a small shop that sells t-shirts. There are

   - 6 sizes
   - 5 colors
   - 20 designs
   - 3 types of fabric
   - 6 languages
   - 2 styles (long or short sleeved).

   Is it reasonable for you to stock 10 of each possible type of t-shirt in your store?

   First we will calculate the number of types of t-shirts. For this, we can just multiply the number of ways to choose each component of the t-shirt design. This gives us

$6 \cdot 5 \cdot 20 \cdot 3 \cdot 6 \cdot 2 = 21600$. So the total number of t-shirts is 216000. It is not hard to see that this is too much to store in a small shop.
What about larger shops? I estimate that about 10 t-shirts can be stored in one cubic foot. So at least 21600 cubic feet are required (and that's not including room for people to walk around in the shop, etc). However, a bit of googling shows that the flagship Macy's is over two million square feet and thus, I assume, at least twenty million cubic feet in volume. So it is plausible that all the t-shirts could comfortably fit in a large store, but I doubt that any store in the world actually stocks so many t-shirts.

5. Suppose that it takes one second to try a password on a computer (I realize that this assumption is not realistic). You want to make sure that nobody can brute-force the password—i.e. determine what the password is simply by sequentially trying all possibilities. In each of the following cases, how long must the password be to make this infeasible? Assume that anything that takes more than a year is not feasible.

(a) The password may contain only digits.

First, it is straightforward to calculate that the number of seconds in a year is about 32000000.
Next, let's calculate the number of passwords with n characters that consist of only digits. There are 10 choices per character, so there are 10^n passwords of length exactly n and $10+10^2+\ldots+10^n$ passwords of length at most n. For simplicity we will assume that only passwords of length exactly n are being considered (although it actually does not make a big difference). So we need n large enough that

$$10^n > 32000000.$$

Either by taking the log of both sides or just by a little experimentation, we can find that we need $n \geq 8$.

(b) The password may contain only uppercase letters.

This is very similar to part (a), but we instead just need n (where once again, n is the number of characters in the password) large enough that

$$26^n > 32000000,$$

which gives us $n \geq 7$.

(c) The password may contain only digits and uppercase letters.

In this case, there are 36 choices for each character so we need

$$36^n > 32000000,$$

giving us $n \geq 5$.

(d) The password may contain either only digits or only uppercase letters, but may not contain both digits and uppercase letters.

There are 26^n passwords of length exactly n containing only uppercase letters and 10^n containing only digits. So in this case there are $26^n + 10^n$ possible passwords of length exactly n. So we need

$$26^n + 10^n > 32000000,$$

giving us $n \geq 7$. Note that this is larger than required by part (c) and in fact no smaller than required in part (b).