Dynamics Worksheet 4 Solutions

For each situation described below, write one or more differential equations that provide a model for it. In each case, also name all relevant functions and constants. **You do not need to solve any of the differential equations you write.**

1. In class, we saw one model of population growth in which the growth rate of a population was proportional to the size of the population times the fraction of the carrying capacity of the environment that is currently "unused" (in other words, if K is the carrying capacity and $P(t)$ is the population at time t then the fraction of the carrying capacity that is unused is $(K - P(t))/K$). This gave us a differential equation called the logistic equation. But what if the population whose growth we are trying to model is some crop that we also wish to harvest? For instance, say we are growing some tasty mushrooms and we harvest the mushrooms at a constant rate. Assume that no mushroom ever dies except when it is harvested. Write a differential equation to express how the size of the mushroom population changes over time.

Solution: Remember that any time you want to write a differential equation to express how to amount of some stuff (in this case, mushrooms) is changing over time, it will look like

$$\text{derivative of the amount of stuff} = (\text{rate in}) - (\text{rate out}).$$

In this case, rate in is the rate of population growth, which we are given. And we are essentially told that the rate out is the constant rate of harvesting. So if we let $P(t)$ denote the size of the mushroom population at time t then we have

$$\frac{dP}{dt} = AP(t)\frac{K - P(t)}{K} - B$$

where A and B are constants (A is the constant of proportionality for the population growth term and B is the constant rate of harvesting).

2. Same scenario as the previous question, but now we are worried about overharvesting the mushrooms, so instead of harvesting them at a constant rate, we harvest them at a rate proportional to the current size of the mushroom population.

Solution: The only thing that has changed from the previous problem is the rate of harvesting, which is now some constant multiple (that's what "proportional" always means) of the size of the mushroom population. So we have

$$\frac{dP}{dt} = AP(t)\frac{K - P(t)}{K} - CP(t)$$

where C is a constant.

3. Same scenario as in the first question, but now suppose that the carrying capacity, rather than being constant, varies over time. In particular, the warmer the weather is, the more

mushrooms the environment can support. So in winter the carrying capacity is small and it increases during the spring, reaching its peak in the summer before shrinking again in the fall. We will model this by supposing that the carrying capacity is a periodic function of time. In particular, let's suppose that the carrying capacity at time t is proportional to $1 + \sin(at)$ where a is some constant (which in this case depends on the length of the seasons).

Solution:

$$\frac{dP}{dt} = AP(t)\frac{D(1 + \sin(at)) - P(t)}{D(1 + \sin(at))} - B$$

where D is some constant.

4. Suppose there are two populations, nematodes (i.e. roundworms) and nematophagous fungi (a carnivorous fungus that eats nematodes). The birth rate for the population of nematodes is proportional to the size of the population. The death rate is proportional to the product of the size of the nematode and fungus populations (since the larger this product is, the higher the chance that some nematode will encounter some of the fungus). On the other hand, the fungus population grows at a rate proportional to this product (since the more the fungus gets to eat, the more it can grow) and decreases at a rate proportional to the current amount of fungus (since then there is more competition for resources). Write two differential equations to express how the sizes of both populations change over time.

Solution: Let $N(t)$ denote the size of the nematode population at time t and $F(t)$ denote the size of the fungus population at time t.

$$\frac{dN}{dt} = aN(t) - bN(t)F(t)$$
$$\frac{dF}{dt} = cN(t)F(t) - dF(t)$$

where a, b, c, and d are constants.

5. You own a widget factory. The cost of producing a widget is proportional to the logarithm of the rate at which you are producing them. In order to avoid running out of money, the rate at which you spend money to produce new widgets is proportional to your total amount of money. Every widget you produce can be sold for some constant price (and let us assume you can sell the widgets immediately upon producing them). Write some differential equations to express how the amount of money you have changes over time.

Comment: In retrospect this is not a very well written question.

Solution: Let $W(t)$ be the number of widgets produced by time t, $M(t)$ be the amount of money you have at time t and $C(t)$ the cost of producing a single widget at time t.

Based on the information given in the problem, we know that

$$\begin{aligned} C(t) &= A\log\left(\frac{dW}{dt}\right) \\ \frac{dM}{dt} &= B\frac{dW}{dt} - C(t)\frac{dW}{dt} \\ \frac{dW}{dt} &= \frac{DM(t)}{C(t)} \end{aligned}$$

where A, B, and D are constants (and in particular, B is the price at which you can sell a widget).