Dynamics Worksheet 3 Solutions

1. Suppose you roll five fair 50 sided dice which each have the numbers 5 through 54 written on them. What is the probability that the sum of the rolls is 50?

Solution: Our sample space, Ω, will be all ordered sequences of five numbers, all of which are between 5 and 54 (each one represents the roll of one of the five dice). Since the dice are fair, all outcomes in Ω are equally likely. Therefore, we can calculate the probability that the sum of the rolls is 50 by dividing the number of such outcomes by the size of Ω.

Since the size of Ω is just 50^5, all that remains is to find the number of outcomes in which the sum of the rolls is 50. We can reformulate this as an equivalent problem involving balls and boxes. Specifically, think of the five dice as five labelled boxes. We have 50 unlabelled balls which we need to throw into the five boxes. The number of balls in each box corresponds to the roll of that die. So each box has to have at least 5 balls and no more than 54 balls. Since there are only 50 balls total, this latter restriction will not concern us here.

In other words, we just need to count the number of ways to throw 50 unlabelled balls into 5 labelled boxes so that every box gets at least 5 balls. So first we throw 5 balls into each box, leaving $50 - 5 \cdot 5 = 25$ balls. We can then distribute the remaining balls in any way and we can count the number of ways to do this using stars-and-bars. There are 25 stars and $5 - 1 = 4$ bars, so the number of ways is:

$$\binom{25+4}{25}.$$

Therefore the probability is

$$\frac{\binom{25+4}{25}}{50^5}.$$

2. Recall the following example from lecture: the rate of infection of some disease depends on the product of the number of people currently infected and the number of people currently uninfected. In lecture, we saw how to write a differential equation to model this scenario. But we ignored the fact that diseases can cause the population to decline (because some people die of the disease). Suppose the death rate in the population is the sum of the regular death rate, which is proportional to the size of the population, plus the rate at which people are killed by the disease, which is proportional to the number of people infected. Suppose also that the birth rate is proportional to the current population size.

   Here are some assumptions you can make: there is no cure for the disease (so once you are infected you remain infected until you die), the infection is not transmitted from mother to baby, and the rate at which infected people die is the sum of the regular death rate and the death rate from the disease.

   (a) Write a pair of differential equations to express how the population size and number of infected people changes over time.

Solution: Let $P(t)$ denote the size of the population at time t and $I(t)$ denote the number of infected people at time t. We want to write a pair of differential equations satisfied by $P(t)$ and $I(t)$. This more or less consists of translating the problem description into the language of differential equations.

In general, if you want to write a differential equation to express how the amount of some substance changes over time, the form it will take is

$$\text{derivative of the amount of the substance} = \text{rate at which the substance is increasing} - \text{rate at which the substance is decreasing}$$

Or, more succinctly:

$$\text{derivative of the amount of the substance} = \text{rate in} - \text{rate out}$$

Here we want to do this for two different "substances": infected people and all people.

For infected people, the rate at which the number of infected people is increasing is simply the rate of infection (because of the assumptions given we do not have to worry about the birth rate). And the rate at which the number of infected people is decreasing is the rate at which infected people die. So we have

$$\frac{dI}{dt} = [C_1 \cdot I(t) \cdot (P(t) - I(t))] - [C_2 \cdot I(t) + C_3 \cdot I(t)]$$

where C_1, C_2, C_3 are the constants of proportionality for the rate of infection, the regular death rate, and the death rate from the disease, respectively.

In case this equation is a little confusing, let's see where each term comes from. The first term, $C_1 \cdot I(t) \cdot (P(t) - I(t))$, is the rate at which new people are infected, which is the proportional to the current number of infected people ($I(t)$) and the current number of uninfected people ($P(t) - I(t)$). The second term, $C_2 \cdot I(t)$ is the rate at which infected people die of things besides the disease. The third term, $C_3 \cdot I(t)$, is the rate at which infected people die of the disease.

For the total population, the rate at which the number of people increases is simply the birth rate and the rate at which the number of people decreases is the death rate, which is the sum of the regular death rate and the death rate from the disease. So we have

$$\frac{dP}{dt} = [C_4 \cdot P(t)] - [C_2 \cdot P(t) + C_3 \cdot I(t)]$$

where C_2 and C_3 are as above and C_4 is the constant of proportionality for the birth rate.

Comment: Notice that the differential equation for $I(t)$ involves $P(t)$ and vice-versa. This is not a problem. When this happens, it is called a system of differential equations and in a few weeks we will learn how to solve a few simple differential equations of this type (though not the one above).

> **Comment:** This problem has been edited slightly from the version on the worksheet I handed out in discussion. Also, in retrospect it is not very well written.

> **Comment:** It is worth pointing out that there are a number of unrealistic assumptions in this problem. As just one of many examples, the birth rate should probably be affected by the number of infected people not just by the total number of people (since people with a disease are usually less likely to have children). The above equations also totally ignore the distribution of ages in the population, which can change over time and can significantly affect the birth and death rate.
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> This does not mean that the above differential equations are worthless, though. In science, differential equations are used to model naturally occurring situations. In all fields, but most of all in biology, it is infeasible or even impossible to account for all the many subtle interactions and effects that occur. So instead you build a simplified model like the one above (which ignores many real-world effects) and hope that it is close enough to reality to make reasonably accurate predictions.
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> So when differential equations are used in biology, there is typically a process of adding or refining assumptions until the differential equation that you get from these assumptions matches reality closely enough to be useful. In this class, you have seen a number of differential equations used in biology, and perhaps in future classes you will see more. It is easy to get the impression that these differential equations are immutable laws handed down from some infallible source. But in truth they are just simplified models of extremely complicated biological systems and they were developed over many years of experimentation and gradual refinement until they provided a good enough approximation of reality to be useful to scientists.

(b) Suppose the birth rate is 0. Do you expect the population to grow or shrink over time? What about the number of infected people? Do you expect the rate at which the population is changing to speed up or slow down over time?

3. A 5 foot tall person is initially standing 3 feet from an 8 foot tall lamp. The person begins to walk forward at a rate of one foot per second. Write a differential equation to express how the length of their shadow changes as they walk.

4. Compute the following indefinite integrals.

$$\int \frac{t^2 - 29t + 5}{(t-4)^2(t^2+3)}\, dt$$

$$\int \frac{t^4 - 5t^3 + 6t^2 - 18}{t^3 - 3t^2}\, dt$$