Dynamics Worksheet 2 Solutions

1. Suppose X is a random variable following the Poisson distribution. In independent trials, you observe the following values of X: 1, 5, 0, 3, 2, 2, 4, 3, 1, 2. Find the 95% confidence interval for the expected value of X.

Solution: Our estimate for the mean, $\widehat{\mu}$, is simply the average of the samples.

$$\widehat{\mu} = \frac{1+5+0+3+2+2+4+3+1+2}{10} = 2.3.$$

Also recall that the expected value and variance of a Poisson distribution are always equal. So our estimate for the variance, $\widehat{\text{Var}}$, is just equal to $\widehat{\mu}$. So the 95% confidence interval is

$$\left(2.3 - 2\sqrt{\frac{2.3}{10}}, 2.3 + 2\sqrt{\frac{2.3}{10}}\right) = (1.34, 3.26)$$

2. When you are falling, your acceleration is the sum of your acceleration due to gravity and your deceleration due to drag. Acceleration due to gravity is constant and deceleration due to drag is proportional to your current velocity. Write a differential equation to express how your velocity changes as you fall. (Hint: acceleration is the derivative of velocity.)

Solution: Let $v(t)$ be your velocity at time t. Then the problem says that

$$\frac{dv}{dt} = g - kv(t)$$

where k is some constant.

3. Newton's second law states that the net force on an object is equal to the acceleration of the object times the mass of an object. Hook's law states that the force a spring exerts on an object is proportional to the distance of the end of the spring from its equilibrium point. Imagine there is a block attached to the end of a spring. Write a differential equation to express how the position of the block is changing (assume the only force on the block comes from the spring).

Solution: Let $x(t)$ be the distance of the block from the equilibrium point of the spring (where $x(t) < 0$ means the block is before the equilibrium point and $x(t) > 0$ means the block is past the equilibrium point). Let $F(t)$ be the total force on the block at time t and let m be the mass of the block. Since acceleration is the second derivative of position, we know that

$$F(t) = m\frac{d^2x}{dt^2}$$
$$F(t) = -kx(t)$$

where k is some positive constant (that depends on how bouncy the spring is). Putting these together gives us

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x(t).$$

4. Find a solution to the following differential equations:

(a) $\frac{dy}{dt} = t^3 + t$

Solution: Integrate both sides to get

$$\int dy = \int t^3 + t\,dt$$
$$y = \frac{t^4}{4} + \frac{t^2}{2} + C$$

where C is some constant.

(b) $\frac{dy}{dt} = \sin(t) + e^t$

Solution: Integrate both sides to get

$$\int dy = \int \sin(t) + e^t\,dt$$
$$y = -\cos(t) + e^t + C$$

where C is some constant.

(c) $\frac{dy}{dt} = e^{-t} + \frac{2t}{t^2-1}$

Solution: Integrate both sides to get

$$\int dy = \int e^{-t} + \frac{2t}{t^2-1}\,dt$$
$$y = -e^{-t} + \int \frac{2t}{t^2-1}\,dt$$

To integrate $\frac{2t}{t^2-1}$ we can just use substitution. Specifically, let $u = t^2 - 1$. So $du = 2tdt$ and

$$\int \frac{2t}{t^2-1}\,dt = \int \frac{2t}{u}\frac{du}{2t} = \int \frac{1}{u}\,du = \ln(u) + C = \ln(t^2-1) + C$$

where C is a constant. So $y = -e^{-t} + \ln(t^2-1) + C$.

(d) $\frac{dy}{dt} + y = e^{-t}$

Solution: We can use an integrating factor to solve this differential equation. In particular, the integrating factor is

$$I(t) = e^{\int dt} = e^t.$$

Multiplying both sides of the equation by $I(t)$ gives us

$$\begin{aligned} I(t)y'(t) + I(t)y(t) &= I(t)e^{-t} \\ \implies \frac{d}{dt}(I(t)y(t)) &= e^t e^{-t} = 1 \end{aligned}$$

Integrating both sides then yields

$$I(t)y(t) = \int dt = t + C$$

(where C is some constant) and so

$$y(t) = \frac{t+C}{I(t)} = \frac{t+C}{e^t}$$

(e) $\frac{dy}{dt} + \frac{3y}{t} = \frac{e^t}{t^3}$

Solution: Integrating factor again. The answer is

$$y(t) = \frac{e^t + C}{t^3}$$

(f) $t\frac{dy}{dt} - 2y = t^2$

Solution: Integrating factor again. But this time we first need to put it in the right form by dividing both sides by t:

$$\frac{dy}{dt} - \frac{2y}{t} = t.$$

The answer is

$$y(t) = \frac{\ln(t) + C}{t^{-2}} = t^2 \ln(t) + Ct^2$$

(g) $t\frac{dy}{dt} - 2y = t^4 \sin t$

Solution: Integrating factor again. First we put it in the right form

$$\frac{dy}{dt} - \frac{2y}{t} = t^3 \sin(t).$$

The integrating factor is $e^{\int -2/t\,dt} = e^{-2\ln(t)} = t^{-2}$. After multiplying both sides by this, we need to integrate $\int t \sin(t)\, dt$. This can be done using integration by parts (note that I think it's very unlikely you'll have to do this on an exam in this class). The final answer is

$$y(t) = \frac{-t\cos(t) + \sin(t) + C}{t^{-2}} = -t^3\cos(t) + t^2\sin(t) + Ct^2.$$