Dynamics Worksheet 1 Solutions

1. Suppose you roll two four-sided dice 100 times each and each time calculate the product of the two rolls. True or false: if you want to perform a χ^2 test to check if the dice are fair based on the data you recorded then the degrees of freedom is 3.

> **Solution:** This is **false**. This is a χ^2 goodness of fit test and there are 9 possible outcomes (the product of two numbers between one and four is always one of 1, 2, 3, 4, 6, 8, 9, 12, or 16) so the degrees of freedom is $9 - 1 = 8$.

2. For each of the following, verify whether the given formula is a solution to the given recurrence relation.

   (a) Recurrence relation:

$$a_n = \sqrt{a_{n-1}^2 + 1};\ a_0 = \sqrt{5}$$

   Formula:

$$a_n = \sqrt{n+5}$$

> **Solution:** First let's recall what it means to be a solution to a recurrence relation. The basic idea is that a recurrence relation tells you how to write down a sequence of numbers. The first few are given by the initial conditions. For the rest, the recurrence relation tells you how to get each number in the sequence if you have already found the previous few numbers. In this case, the sequence described by the recurrence relation is
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> $$\begin{aligned} &\sqrt{5}, \\ \sqrt{\left(\sqrt{5}\right)^2 + 1} &= \sqrt{6}, \\ \sqrt{\left(\sqrt{6}\right)^2 + 1} &= \sqrt{7}, \\ \sqrt{\left(\sqrt{7}\right)^2 + 1} &= \sqrt{8}, \\ \sqrt{\left(\sqrt{8}\right)^2 + 1} &= \sqrt{9}, \\ &\vdots \end{aligned}$$
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> A formula is a solution to a recurrence relation if it gives the same sequence of numbers (or, if there are no initial conditions, if it gives a sequence of numbers that satisfy the recurrence relation). In this case, the given formula gives us the following

sequence of numbers

$$\sqrt{0+5} = \sqrt{5},$$
$$\sqrt{1+5} = \sqrt{6},$$
$$\sqrt{2+5} = \sqrt{7},$$
$$\sqrt{3+5} = \sqrt{8},$$
$$\sqrt{4+5} = \sqrt{9},$$
$$\vdots$$

So far, it looks the same as the sequence given by the recurrence relation. But it is not enough to check that the first few terms are the same. We need to make sure that *all* the terms are the same. We can do this by checking that the formula matches the initial conditions and that for any n, the formula satisfies the recurrence relation.

Initial conditions: we need to check that the given formula for a_n satisfies $a_0 = \sqrt{5}$, which it does.

Recurrence relation: we need to check that for all n, the given formula for a_n satisfies $a_n = \sqrt{a_{n-1}^2 + 1}$. We can do this by plugging the formula into this recurrence relation to get:

$$\sqrt{a_{n-1}^2 + 1} = \sqrt{\left(\sqrt{(n-1)+5}\right)^2 + 1} = \sqrt{(n-1)+5+1} = \sqrt{n+5}$$

Since this matches the formula for a_n, the formula does satisfy the recurrence relation. Therefore the given formula $\boxed{\text{is a solution}}$.

(b) Recurrence relation:

$$a_n = 2^{a_{n-1}} + a_{n-2};\ a_0 = 1,\ a_1 = 2$$

Formula:

$$a_n = \sqrt{n+5}$$

Solution: We won't go into nearly so much detail as in the solution to part (a). Just as before, we need to check that the formula satisfies the initial conditions and the recurrence relation.

Initial conditions: the formula for a_0 is $\sqrt{0+5} = \sqrt{5}$, which is not equal to 1. So the formula does not satisfy the initial conditions. This means the formula is $\boxed{\text{not a solution}}$. So we could stop right here, but let's see if it even satisfies the other initial condition or the recurrence relation.

The other initial condition is that $a_1 = 2$. But the formula gives us $a_1 = \sqrt{6}$. So the formula also doesn't satisfy the other initial condition.

Recurrence relation: plugging the formula into the recurrence relation gives

$$2^{\sqrt{(n-1)+5}} + \sqrt{(n-2)+5}$$

It seems unlikely that this is equal to the formula for a_n, $\sqrt{n+5}$, but maybe that seems hard to show. But recall that to satisfy the recurrence relation, the formula needs to match this for *all* values of n. So to show the formula doesn't satisfy the recurrence relation, it is enough to show that it fails to hold for at least one value of n. For instance, taking $n = 5$ we have

$$2^{\sqrt{(5-1)+5}} + \sqrt{(5-2)+5} = 2^3 + \sqrt{8}$$

which it is easy to verify is not equal to $\sqrt{5+5} = \sqrt{10}$. So the formula does not satisfy the recurrence relation.

(c) Recurrence relation:

$$a_n = 2a_{n-1} + 3a_{n-2};\ a_0 = 0,\ a_1 = 4$$

Formula:

$$a_n = n^2 + 3n$$

Solution: Initial conditions:

$$0^2 + 3 \cdot 0 = 0$$
$$1^2 + 3 \cdot 1 = 4$$

So the formula does satisfy the initial conditions.
Recurrence relation:

$$\begin{aligned} 2((n-1)^2 + 3(n-1)) + 3((n-2)^2 + 3(n-2)) &= 2(n^2 - 2n + 1 + 3n - 3) + 3(n^2 - 4n + 4 + 3n - 6) \\ &= 5n^2 - n - 10 \end{aligned}$$

For most values of n, this is not equal to $n^2 + 3n$ (for instance, try plugging in $n = 2$). So the formula does not satisfy the recurrence relation and therefore is **not a solution**.

(d) Recurrence relation:

$$a_n = 2a_{n-1} + 3a_{n-2};\ a_0 = 0,\ a_1 = 4$$

Formula:

$$a_n = 3^n$$

Solution: This is **not a solution** since it does not satisfy the initial conditions. However, it does satisfy the recurrence relation.

(e) Recurrence relation:

$$a_n = 2a_{n-1} + 3a_{n-2};\ a_0 = 0,\ a_1 = 4$$

Formula:

$$a_n = 3^n - (-1)^n$$

Solution: This one $\boxed{\text{is a solution}}$. I will just show that it satisfies the recurrence relation, though on a test or quiz you should also show that it satisfies the initial conditions. Plugging the formula into the recurrence relation gives

$$\begin{aligned} 2(3^{n-1} - (-1)^{n-1}) + 3(3^{n-2} - (-1)^{n-2}) &= 3^{n-2}(2 \cdot 3 + 3) + (-1)^{n-2}(2 - 3) \\ &= 3^{n-2} \cdot 9 - (-1)^{n-2} \\ &= 3^n - (-1)^n \end{aligned}$$

3. Find a formula for the n^{th} Fibonacci number. Recall that the first two Fibonacci numbers (i.e. the 0^{th} and 1^{st} Fibonacci numbers) are both 1 and that to get the next Fibonacci number, you add the previous two. First formulate this as a recurrence relation and then try to solve it.

Solution: The first few terms of the Fibonacci sequence are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$. It can be expressed as the following recurrence relation:

$$a_n = a_{n-1} + a_{n-2};\ a_0 = 1,\ a_1 = 1.$$

(The first part just says that each term is the sum of the previous two terms and the second part just says that the first two terms are both 1.)

Luckily for us, this recurrence relation is of the form that we learned how to solve in class. We begin by writing the characteristic equation:

$$\lambda^2 = \lambda + 1.$$

We can find solutions to this using the formula for roots of a quadratic polynomial:

$$0 = \lambda^2 - \lambda - 1 \implies \lambda = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

So the general solution to the recurrence relation (ignoring for now the initial conditions) is

$$a_n = C_1 \left(\frac{1 + \sqrt{5}}{2}\right)^n + C_2 \left(\frac{1 - \sqrt{5}}{2}\right)^n$$

where C_1 and C_2 can be any constants.

Finally, we need to find appropriate values of C_1 and C_2 to match the initial conditions. To match the initial conditions, we need our formula for a_n to be equal to 1 when n is 0 and when n is 1. Plugging these values of n into our formula we get

$$\begin{aligned} 1 &= C_1 \left(\frac{1 + \sqrt{5}}{2}\right)^0 + C_2 \left(\frac{1 - \sqrt{5}}{2}\right)^0 = C_1 + C_2 \\ 1 &= C_1 \left(\frac{1 + \sqrt{5}}{2}\right)^1 + C_2 \left(\frac{1 - \sqrt{5}}{2}\right)^1 = C_1 \left(\frac{1 + \sqrt{5}}{2}\right) + C_2 \left(\frac{1 - \sqrt{5}}{2}\right) \end{aligned}$$

We now have two linear equations in two variables, which we can solve for C_1 and C_2. This gives us

$$C_1 = \frac{5+\sqrt{5}}{10},\ C_2 = \frac{5-\sqrt{5}}{10}.$$

So the formula for the n^{th} Fibonacci number is

$$\frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Comment: The value $\frac{1+\sqrt{5}}{2}$ is known as the "golden ratio" and the ancient Greeks (among others) were pretty into it.

Comment: Recall that our reason for writing down the characteristic equation was as follows: we make the inspired guess that there are solutions of the form λ^n where λ is some constant. If there are such solutions, they would need to satisfy the recurrence relation. I.e. they would need to satisfy

$$\lambda^n = \lambda^{n-1} + \lambda^{n-2}$$

for all $n > 1$. But dividing both sides by λ^{n-2} this gives us

$$\lambda^2 = \lambda + 1$$

which is exactly the characteristic equation.

If the recurrence relation were a bit different, say $b_n = 2b_{n-1} - 3b_{n-2} + 7b_{n-4}$ then the characteristic equation would change accordingly, in this case to $\lambda^4 = 2\lambda^3 - 3\lambda^2 + 7$.

Finally, why was our general solution $C_1\lambda_1^n + C_2\lambda_2^n$ where λ_1 and λ_2 were the two solutions to the characteristic equation? This was because for recurrence relations of a very specific form (linear, constant coefficient and homogeneous) the sum of any two solutions and any constant multiple of any solution are also solutions.

4. Find a solution to the following recurrence relation:

$$a_n = n \cdot a_{n-1};\ a_0 = 1$$

Solution: This recurrence relation is not of the type that we learned how to solve in class. So let's just start by writing out the first few terms and seeing if we can spot any patterns.

$$\begin{aligned}
a_0 &= 1\\
a_1 &= 1 \cdot a_0 = 1 \cdot 1 = 1\\
a_2 &= 2 \cdot a_1 = 2 \cdot 1 = 2\\
a_3 &= 3 \cdot a_2 = 3 \cdot 2 = 6\\
a_4 &= 4 \cdot a_3 = 4 \cdot 6 = 24\\
a_5 &= 5 \cdot a_4 = 5 \cdot 24 = 120
\end{aligned}$$

At this point, you might notice that these numbers look familiar. But if not, it can often help to *not* simplify things. If we write out the same first six terms without simplifying things, we get this:

$$\begin{aligned}
a_0 &= 1\\
a_1 &= 1 \cdot a_0 = 1 \cdot 1\\
a_2 &= 2 \cdot a_1 = 2 \cdot 1 \cdot 1\\
a_3 &= 3 \cdot a_2 = 3 \cdot 2 \cdot 1 \cdot 1\\
a_4 &= 4 \cdot a_3 = 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1\\
a_5 &= 5 \cdot a_4 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1
\end{aligned}$$

Now the pattern is more apparent and we are led to guess that a solution might be

$$a_n = n \cdot (n-1) \cdots 2 \cdot 1 \cdot 1 = n!$$

However, **just because this formula looks like it matches the pattern in the first few terms, that does not automatically mean it is a solution to the recurrence relation—we need to actually prove that it is a solution**. To prove it is a solution, it is enough to show that it satisfies the initial condition and the recurrence relation.

Initial condition:

$$a_0 = 0! = 1.$$

So the formula satisfies the initial condition.

Recurrence relation:

$$n \cdot a_{n-1} = n \cdot (n-1)! = n!$$

Since this is equal to the formula for a_n, the formula satisfies the recurrence relation.

5. **Challenge Question:** Look up the rules to the Tower of Hanoi game and find a formula for the least number of moves it takes to win the game when there are n disks.