# Discrete Probability and Review Worksheet 6 Solutions

1. Suppose there are 12 people in a room. Show that you can choose two groups of people in the room such that the sum of ages (in years) of both groups is the same. Note: the groups should not overlap but they may contain just one person each.

**Solution:** There are $2^{12}-1$ nonempty subsets of the 12 people. And since (it is believed) nobody has lived past 130 years, the sum of the ages of everyone in a subset of the 12 people must be a number between 0 and $12 \cdot 130 = 1560$. Since $2^{12} - 1 > 1560$, by the pigeonhole principle there are two distinct subsets of the 12 people whose members' ages sum to the same value.

To get two *disjoint* subsets, as the problem requires, just remove anybody who is in both subsets. More formally, let $A$ and $B$ be two subsets of the twelve people such that the sums of the ages are the same. Then the sums of the ages of the people in $A-(A \cap B)$ and $B-(A\cap B)$ are also the same.

2. How many anagrams does "ouroboros" have?

**Solution:**

$$\binom{9}{4}\binom{5}{2}\binom{3}{1}\binom{2}{1}\binom{1}{1} = \frac{9!}{4!2!} = \binom{9}{4,\ 2,\ 1,\ 1,\ 1}$$

3. Suppose you roll a fair 4-sided die 7 times in a row. What is the probability that all 4 numbers are rolled at least once?

**Solution:** Let's let our sample space, $\Omega$, be the set of all sequences of 7 numbers between 1 and 4. Since the die is fair, each outcome in $\Omega$ is equally likely. Note that $|\Omega| = 4^7$. Let $A$ be the event that each number is rolled at least once—i.e. the set of sequences of 7 numbers between 1 and 4 in which each number between 1 and 4 appears at least once. We need to calculate $|A|$. It is not obvious how to do this directly, so we will calculate $|A^c|$ instead. To do this we will use inclusion-exclusion. Let $A_i$ be the event that $i$ is not rolled, where $i$ is any number between 1 and 4. Then $A^c = A_1 \cup A_2 \cup A_3 \cup A_4$. We have

$$\begin{aligned}
|A^c| = {} & |A_1| + |A_2| + |A_3| + |A_4| \\
& - |A_1 \cap A_2| - \ldots - |A_3 \cap A_4| \\
& + |A_1 \cap A_2 \cap A_3| + \ldots + |A_2 \cap A_3 \cap A_4| \\
& - |A_1 \cap A_2 \cap A_3 \cap A_4| \\
= {} & 4(3^7) - \binom{4}{2} 2^7 + 4(1^7) - 0 \\
= {} & 4(3^7) - 6(2^7) + 4.
\end{aligned}$$

Therefore,

$$P(A) = 1 - P(A^c) = 1 - \frac{|A^c|}{|\Omega|} = 1 - \frac{4(3^7) - 6(2^7) + 4}{4^7}.$$

4. Suppose you and your three friends find 100 identical marbles on the ground.

(a) How many ways are there to divide the marbles between you and your friends?

**Solution:** This is the same as asking how many ways there are to put 100 identical balls in 4 distinguishable boxes, so we can use stars and bars to get

$$\binom{100+4-1}{100} = \binom{103}{100}.$$

(b) How many ways are there to divide the marbles if everybody has to get at least three marbles?

**Solution:** We will use stars and bars again, but first subtract 12 marbles from the total number (because everybody first gets three marbles and then we distribute the rest). So the answer is

$$\binom{100-12+4-1}{100-12} = \binom{91}{88}.$$

(c) How many ways are there to divide the marbles if nobody can get more than 30 marbles?

**Solution:** It is not obvious how to calculate this directly, so we will first try to calculate the number of ways in which somebody gets more than 30 marbles. Let $A$ denote the set of ways to distribute the marbles in which at least one person gets more than 30. To find $|A|$, we will use inclusion-exclusion. Let's number you and your friends 1 through 4. Let $A_i$ be the set of ways to distribute the marbles in which person $i$ gets more than 30 marbles. Then we have

$$\begin{aligned}
|A| &= |A_1| + |A_2| + |A_3| + |A_4| \\
&\quad - |A_1 \cap A_2| - \ldots - |A_3 \cap A_4| \\
&\quad + |A_1 \cap A_2 \cap A_3| + \ldots + |A_2 \cap A_3 \cap A_4| \\
&\quad - |A_1 \cap A_2 \cap A_3 \cap A_4| \\
&= 4\binom{100-31+4-1}{100-31} - \binom{4}{2}\binom{100-62+4-1}{100-62} \\
&\quad + 4\binom{100-93+4-1}{100-93} - 0 \\
&= 4\binom{72}{69} - 6\binom{41}{38} + 4\binom{10}{7}.
\end{aligned}$$

Therefore, the number of ways in which everybody gets at most 30 marbles is

$$\binom{103}{100} - |A| = \boxed{\binom{103}{100} - \left(4\binom{72}{69} - 6\binom{41}{38} + 4\binom{10}{7}\right)}.$$

There's also another way to solve this problem which requires a bit less calculation. Imagine we first give 30 marbles to everyone. But now we have given out 120 marbles when we were only supposed to give out 100. So we decide how to take away 20 marbles total from the 4 people. This ensures that nobody gets more than 30 marbles. Also, since 20 is less than 30, we don't have to worry about taking away *more than* 30 marbles from anyone (which would leave them with less than 0 marbles, which doesn't make any sense). Using stars and bars, there are $\boxed{\binom{20+4-1}{4-1}}$ ways to do this. You can use a calculator to verify that this matches the answer we got with the other method.