Discrete Probability and Review Worksheet 5 Solutions

1. How many ways are there to choose a password of length 20 that contains only lowercase letters or digits and that has exactly three a's and at least one b?

> **Solution:** Instead of trying to calculate this directly we will first find the number of passwords that have exactly three a's and the number of passwords that have exactly three a's and no b's. We will subtract this second number from the first to arrive at the final answer.
>
> The number of passwords of length 20 that contain only lowercase letters or digits and that have exactly three a's is $\binom{20}{3}35^{17}$: first we choose which three of the twenty characters in the password will be a's and then we fill in the other characters with any other lowercase letters or digits besides a (since we can't have more than 3 a's).
>
> By essentially the same reasoning as above, the number of passwords of length 20 that contain only lowercase letters or digits and that have exactly three a's and *no* b's is $\binom{20}{3}34^{17}$.
>
> So the final answer is $\boxed{\binom{20}{3}35^{17} - \binom{20}{3}34^{17}}$.

2. What is the sum of the coefficients of $(x_1 + x_2 + \ldots + x_k)^n$?

> **Solution:** This is equivalent to the number of ways to pick one term from each of the n sums $x_1 + x_2 + \ldots + x_k$, which is $\boxed{k^n}$.

3. Suppose we choose a subset of $\{1, 2, \ldots, n\}$ by flipping a fair coin to choose whether each element is in the subset or not. Define a random variable X to be the size of the resulting subset. What is $E[X]$?

> **Solution:** This is an example of a binomial distribution with n trials (one for each element of the set $\{1, 2, \ldots, n\}$) and success probability $1/2$ (the chance that each element gets into the subset). So the expected value is $\boxed{n/2}$. You can also calculate this using linearity of expectation (which is how we found the expected value of the binomial distribution in class).

4. Suppose you repeatedly roll a fair six-sided die until the *second* time you get a six. Define the random variable X to be the number of times you roll the die. Find a formula for $P(X = k)$.

> **Solution:** To have exactly k rolls of the die, we need to roll the die $k - 1$ times and get exactly one six and then roll it once more and get a six.
>
> Let's first focus on the probability of rolling the die $k - 1$ times and getting exactly one six. There are $k - 1$ ways for this to happen: either the first roll is a six and the next $k - 2$

are not, or the first roll is not a six, the second one is and the next $k-3$ are not, and so on. Each of this ways occurs with probability $(1/6)(5/6)^{k-2}$. So the total probability of rolling $k-1$ times and getting exactly one six is

$$(k-1)\frac{1}{6}\left(\frac{5}{6}\right)^{k-2}.$$

The probability of getting a six on the k^{th} roll is of course 1/6. So the probability of rolling exactly k times is

$$(k-1)\left(\frac{1}{6}\right)^{2}\left(\frac{5}{6}\right)^{k-2}.$$

5. Suppose we roll a fair six-sided die twice. Define three random variables as follows: X is the result of the first roll, Y is the sum of the two rolls, and Z is always -2.

(a) Find the probability mass function for each random variable.

Solution: For X:

- $P(X=1)=1/6$
- $P(X=2)=1/6$
- $P(X=3)=1/6$
- $P(X=4)=1/6$
- $P(X=5)=1/6$
- $P(X=6)=1/6$

For Y:

- $P(Y=2)=1/36$
- $P(Y=3)=2/36$
- ...
- $P(Y=7)=6/36$
- $P(Y=8)=5/36$
- ...
- $P(Y=12)=1/36$

We can also write this as follows:

$$P(Y=k)=\begin{cases}\frac{k-1}{36} & \text{if } 2\le k\le 7\\ \frac{13-k}{36} & \text{if } 8\le k\le 12\\ 0 & \text{otherwise}\end{cases}$$

For Z, $P(Z=-2)=1$.

(b) Find the expected value of each random variable.

Solution: For X:

$$E[X] = \sum_{k=1}^{6} k \cdot P(X = k) = \sum_{k=1}^{6} k \cdot \frac{1}{6} = 3.5$$

For Y:

$$\begin{aligned} E[Y] &= \sum_{k=2}^{12} k \cdot P(Y = k) \\ &= 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + \ldots + 7 \cdot \frac{6}{36} + 8 \cdot \frac{5}{36} + \ldots + 12 \cdot \frac{1}{36} \\ &= 7 \end{aligned}$$

For Z, $E[Z] = -2$.

(c) Find the variance of each random variable.

Solution: For X:

$$\begin{aligned} \text{Var}[X] &= E[(X - E[X])^2] \\ &= E[(X - 3.5)^2] \\ &= \sum_{k=1}^{6} (k - 3.5)^2 \frac{1}{6} \\ &= \frac{17.5}{6} \approx 2.917 \end{aligned}$$

We can also calculate this using the fact that $\text{Var}[X] = E[X^2] - E[X]^2$:

$$\begin{aligned} \text{Var}[X] &= E[X^2] - E[X]^2 \\ &= \sum_{k=1}^{6} k^2 P(X = k) - 3.5^2 \\ &= \sum_{k=1}^{6} k^2 \frac{1}{6} - 3.5^2 \\ &= \frac{91}{6} - 3.5^2 \end{aligned}$$

For Y we will also use the fact that $\text{Var}[Y] = E[Y^2] - E[Y]^2$:

$$\begin{aligned} \text{Var}[Y] &= E[Y^2] - E[Y]^2 \\ &= \sum_{k=2}^{1} 2k^2 P(Y = k) - 7^2 \\ &= \frac{1974}{36} - 7^2 \approx 5.83 \end{aligned}$$

For Z, since Z is constant, the variance is 0.

6. **Challenge Question:** Suppose you are given a biased coin for which the probability of heads is some unknown constant p. How can you use this coin to simulate flipping a *fair* coin? In your scheme, how many times on average do you need to flip the biased coin to simulate one flip of a fair coin?