Discrete Probability Worksheet 4 Solutions

1. You roll two fair six-sided dice. Let X be the random variable that is the sum of the rolls. Write down the probability mass function for X.

Solution: Recall that the probability mass function of a random variable is just the probability that the random variable takes each value in its range. In this case, we have

$$\begin{aligned}
P(X = 2) &= \frac{1}{36}\\
P(X = 3) &= \frac{2}{36}\\
P(X = 4) &= \frac{3}{36}\\
P(X = 5) &= \frac{4}{36}\\
P(X = 6) &= \frac{5}{36}\\
P(X = 7) &= \frac{6}{36}\\
P(X = 8) &= \frac{5}{36}\\
P(X = 9) &= \frac{4}{36}\\
P(X = 10) &= \frac{3}{36}\\
P(X = 11) &= \frac{2}{36}\\
P(X = 12) &= \frac{1}{36}
\end{aligned}$$

2. In Zurich, the trams cost 2 francs per ride. It is possible to ride the trams without a pass, but if there is an inspection you will be caught and have to pay a fine of 100 francs.

   (a) How frequently should the inspections be so that people are better off buying the passes than riding illegally?

Solution: Let's let p be the frequency of inspections (i.e. the probability that there is an inspection on any given tram ride) and we will figure what p needs to be. Consider a single tram ride where you don't buy a pass. Let the random variable X be the amount of money you pay (in francs). We want p to be large enough so that $E[X]$ is larger than 2, in otherv words, the average amount of money people pay in fines when they don't have a pass should be larger than the cost of the pass itself.

With probability p, there is an inspection and you need to pay 100 francs in fines. With probability $1-p$ there is no inspection and you don't need to pay anything. So $E[X] = 100 \cdot p + 0 \cdot (1-p) = 100p$. We need $100p > 2$, or in other words $p > \frac{1}{50}$. So there should be an inspection at least once every fifty rides.

(b) Now suppose that the second time you get caught, you are executed (so after you get caught once you will make sure to always buy a pass). Suppose the probability of an inspection on any given ride is $\frac{1}{200}$. What is the expected amount of money you save by not buying passes on all the rides up until you get caught for the first time (include the 100 franc fine in your calculation)? What is the name for this type of distribution?

> **Solution:** Let X be the number of tram rides before you get caught for the first time. On each of these tram rides we save 2 francs by not buying a pass. On the tram ride when you get caught for the first time, you lose 98 francs (you lose 100 from the fine but save 2 by not buying a pass). So the total amount of money you save is $2 \cdot E[X] - 98$.
>
> Now notice that X is just a geometrically distributed random variable with parameter $\frac{1}{200}$ (we are assuming here that whether or not there is an inspection on two different tram rides are independent events). So we have
>
> $$E[X] = \frac{1 - \frac{1}{200}}{\frac{1}{200}} = 199.$$
>
> Therefore we save $\boxed{199 - 98 = 101}$ francs.

3. There are n people who each have their own hat. You take all the hats and randomly rearrange them. Let the random variable X be the number of people who get their own hat back. What is $E[X]$?

> **Solution:** We will use the magic of linearity of expectation. For $1 \leq i \leq n$, let X_i be the random variable defined as follows:
>
> $$X_i = \begin{cases} 1 & \text{if person } i \text{ gets their own hat back} \\ 0 & \text{otherwise} \end{cases}$$
>
> Note that $X = X_1 + X_2 + \ldots + X_n$. So by linearity of expectation
>
> $$E[X] = E[X_1 + X_2 + \ldots + X_n] = E[X_1] + E[X_2] + \ldots + E[X_n].$$
>
> All that remains is to calculate $E[X_i]$. Person i receives their hat with probability $\frac{(n-1)!}{n!} = \frac{1}{n}$ so $E[X_i] = 1 \cdot \frac{1}{n} + 0 \cdot \frac{n-1}{n} = \frac{1}{n}$. Therefore
>
> $$E[X] = n\left(\frac{1}{n}\right) = 1.$$

4. Consider the scenario described in the previous problem when there are just two people. What is $\mathrm{Var}[X]$?

Solution: If there are only two people then either both get back their own hat or neither gets back their own hat. And each of these two possibilities happens with probability 1/2. So we can calculate $E[X^2]$ as follows:

$$E[X^2] = 0 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{2} = 2.$$

Therefore

$$\mathrm{Var}[X] = E[X^2] - E[X]^2 = 2 - 1^2 = 1.$$

Comment: Actually, you can show that even for n people, $\mathrm{Var}[X] = 1$. Showing that doesn't use anything we haven't learned, but it is a little tricky.

5. If X is a random variable and $\mathrm{Var}[X] = 0$, what can you say about X?

Solution: Recall that $\mathrm{Var}[X] = E[(X - E[X])^2]$. Since $(X - E[X])^2$ is never negative, if it is ever positive then its expected value will also be positive. Therefore, if $\mathrm{Var}[X] = 0$ then X must always be equal to $E[X]$. In other words, X must be constant.

6. Suppose you roll 20 fair 6-sided dice. Let the random variable X be the sum of the rolls.
   (a) What is $E[X]$?

Solution: For each $i \leq 20$ let the random variable X_i be the value of the i^{th} die. Then $X = X_1 + X_2 + \ldots + X_{20}$. So by linearity of expectation:

$$E[X] = E[X_1] + E[X_2] + \ldots + E[X_{20}].$$

Now we directly calculate $E[X_i]$:

$$E[X_i] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{7}{2}.$$

Therefore $E[X] = 20 \cdot \frac{7}{2} = 70$.

   (b) What is $\mathrm{Var}[X]$?

Solution: Note that the random variables X_i from the solution to part (a) are actually independent random variables (**warning: most of the time when you have a sum of random variables, they are not independent and this trick will not work**). Therefore

$$\mathrm{Var}[X] = \mathrm{Var}[X_1] + \mathrm{Var}[X_2] + \ldots + \mathrm{Var}[X_{20}].$$

Now we calculate $\mathrm{Var}[X_i]$:

$$\begin{aligned}
\mathrm{Var}[X_i] &= E[X_i^2] - E[X_i]^2 \\
&= \left(1 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 9 \cdot \frac{1}{6} + 16 \cdot \frac{1}{6} + 25 \cdot \frac{1}{6} + 36 \cdot \frac{1}{6}\right) - \left(\frac{7}{2}\right)^2 \\
&= \frac{91}{6} - \frac{49}{4} \\
&= \frac{35}{12}.
\end{aligned}$$

Therefore $\mathrm{Var}[X] = 20 \cdot \frac{35}{12} = \frac{175}{3}$.