Math 10B Midterm 1 Review

1. Suppose that you have n employees and need to choose some of them to receive a promotion. In each of the following scenarios, how many ways are there to choose which employees receive a promotion?

   (a) Suppose you can choose any number of employees to receive a promotion.

   For each employee we must choose if they receive the promotion or not. This gives us n choices to make, with two possibilities for each. So there are 2^n ways.

   (b) Exactly 5 employees must receive a promotion.

   $$\binom{n}{5}.$$

   (c) Any number of employees can receive a promotion, but at least one of the employees Alan, Kim, and Cassandra must receive a promotion.

   The number of ways for none of Alan, Kim, and Cassandra to receive a promotion is 2^{n-3} since we just have to choose which of the other $n-3$ employees receive promotions. So there are $2^n - 2^{n-3}$ ways in which at least one of the three receives a promotion.

   (d) Any number of employees can receive a promotion, but at *most* one of the employees Alan, Kim, and Cassandra must receive a promotion.

   There are 4 possibilities: none of the three receive a promotion, only Alan does, only Kim does, or only Cassandra does. In each of these, there are 2^{n-3} ways to choose which of the other employees receive promotions. So there are $2^{n-3} + 2^{n-3} + 2^{n-3} + 2^{n-3} = 4 \cdot 2^{n-3}$ ways.

2. Suppose there are 12 people in a room. Show that you can choose two groups of people in the room such that the sum of ages (in years) in both groups is the same.

   There are 2^{12} subsets of the 12 people. And since (it is believed) nobody has lived past 130 years, the sum of the ages of everyone in a subset of the 12 people must be a number between 0 and $12 \cdot 130 = 1560$. Since $2^{12} > 1560$, by the pigeonhole principle there are two distinct subsets of the 12 people whose members' ages sum to the same value.

3. Let x be any positive real number. Show that for all $n \geq 2$, $(1+x)^n > 1 + nx$.

The proof is by induction on n.
The base case is $n = 2$. Note that $(1+x)^2 = 1 + 2x + x^2$, which is greater than $1 + 2x$ because $x^2 > 0$.
For the inductive case, assume that $(1+x)^k > 1 + kx$. We will show that $(1+x)^{k+1} > 1 + (k+1)x$. Indeed,

$$\begin{aligned}
(1+x)^{k+1} &= (1+x)(1+x)^k \\
&> (1+x)(1+kx) && \text{by the inductive assumption} \\
&= 1 + (k+1)x + kx^2 \\
&> 1 + (k+1)x && \text{because } x^2 > 0 \text{ and } k > 0.
\end{aligned}$$

4. How many anagrams does the word "ouroboros" have?

$$\binom{9}{4,2,1,1,1} = 3\binom{9}{4}\binom{5}{2}.$$

5. Suppose you roll a fair 4-sided die 7 times in a row. What is the probability that all 4 numbers are rolled at least once?

Let's let our sample space, Ω, be the set of all sequences of 7 numbers between 1 and 4. Since the die is fair, each outcome in Ω is equally likely. Note that $|\Omega| = 4^7$. Let A be the event that each number is rolled at least once—i.e. the set of sequences of 7 numbers between 1 and 4 in which each number between 1 and 4 appears at least once. We need to calculate $|A|$. It is not obvious how to do this directly, so we will calculate $|A^c|$ instead. To do this we will use inclusion-exclusion. Let A_i be the event that i is not rolled, where i is any number between 1 and 4. Then $A^c = A_1 \cup A_2 \cup A_3 \cup A_4$. We have

$$\begin{aligned}
|A^c| &= |A_1| + |A_2| + |A_3| + |A_4| \\
&\quad - |A_1 \cap A_2| - \ldots - |A_3 \cap A_4| \\
&\quad + |A_1 \cap A_2 \cap A_3| + \ldots + |A_2 \cap A_3 \cap A_4| \\
&\quad - |A_1 \cap A_2 \cap A_3 \cap A_4| \\
&= 4(3^7) - \binom{4}{2}2^7 + 4(1^7) - 0 \\
&= 4(3^7) - 6(2^7) + 4.
\end{aligned}$$

Therefore,

$$P(A) = \frac{|A|}{|\Omega|} = 1 - \frac{|A^c|}{|\Omega|} = 1 - \frac{4(3^7) - 6(2^7) + 4}{4^7}.$$

6. What is the coefficient of x^6y^7 in $(3x^2 - y)^{10}$?

$$\binom{10}{7}3^3(-1)^7 = -27\binom{10}{7}.$$

7. Suppose you and three of your friends find 100 identical marbles on the ground.

   (a) How many ways are there to divide the marbles between you and your friends?

   This is the same as asking how many ways there are to put 100 identical balls in 4 distinguishable boxes, so we can use stars and bars to get

   $$\binom{100+4-1}{100} = \binom{103}{100}.$$

   (b) How many ways are there to divide the marbles if everybody has to get at least three marbles?

   We will use stars and bars again, but first subtract 12 marbles from the total number (because everybody first gets three marbles and then we distribute the rest). So the answer is

   $$\binom{100-12+4-1}{100-12} = \binom{91}{88}.$$

   (c) How many ways are there to divide the marbles if nobody can get more than 30 marbles?

   It is not obvious how to calculate this directly, so we will first try to calculate the number of ways in which somebody gets more than 30 marbles. Let A denote the set of ways to distribute the marbles in which at least one person gets more than 30. To find $|A|$, we will use inclusion-exclusion. Let's number you and your friends 1 through 4. Let A_i be the set of ways to distribute the marbles in which person i gets more than 30 marbles. Then we have

   $$\begin{aligned}
   |A| &= |A_1| + |A_2| + |A_3| + |A_4| \\
   &\quad - |A_1 \cap A_2| - \ldots - |A_3 \cap A_4| \\
   &\quad + |A_1 \cap A_2 \cap A_3| + \ldots + |A_2 \cap A_3 \cap A_4| \\
   &\quad - |A_1 \cap A_2 \cap A_3 \cap A_4| \\
   &= 4\binom{100-31+4-1}{100-31} - \binom{4}{2}\binom{100-62+4-1}{100-62} \\
   &\quad + 4\binom{100-93+4-1}{100-93} - 0 \\
   &= 4\binom{72}{69} - 6\binom{41}{38} + 4\binom{10}{7}.
   \end{aligned}$$

Therefore, the number of ways in which everybody gets at most 30 marbles is

$$\binom{103}{100} - |A| = \binom{103}{100} - \left(4\binom{72}{69} - 6\binom{41}{38} + 4\binom{10}{7}\right).$$