1 Systems of differential equations

Problem 1
Solve $x' = Ax$, and find the fundamental matrix $X(t)$, where:

$$A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
8 & -14 & 7
\end{bmatrix}$$

Problem 2
Solve $x' = Ax$, where:

$$A = \begin{bmatrix}
-1 & -2 & 0 \\
8 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}$$

Problem 3
Solve $x' = Ax$, where:

$$A = \begin{bmatrix}
5 & -3 \\
3 & -1
\end{bmatrix}$$

2 Grünbaum’s coupled harmonic oscillator

Problem 4
Assume you’re given a coupled mass-spring system with $N = 2$, $m_1 = m_2 = 1$, $k_1 = k_2 = k_3 = 1$. Find the proper frequencies and the proper modes of the system.
3 Partial differential equations and Fourier series

Problem 5
Find the Fourier cosine series of \( f(x) = x^2 \) on \((0, \pi)\)

Problem 6
Find the solution of the following heat equation:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= 4 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0 \\
\frac{\partial u}{\partial x}(0, t) &= \frac{\partial u}{\partial x}(\pi, t) = 0, \quad t > 0 \\
u(x, 0) &= x, \quad 0 < x < \pi
\end{align*}
\]

Problem 7
Find the solution of the following wave equation:

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} &= 9 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0 \\
u(0, t) &= u(1, t) = 0, \quad t > 0 \\
u(x, 0) &= 3 \sin(3\pi x), \quad 0 < x < 1 \\
\frac{\partial u}{\partial t}(x, 0) &= 5 \sin(4\pi x), \quad 0 < x < 1
\end{align*}
\]

4 Higher-order differential equations

Problem 8
Are the functions \( xe^x, x^2 e^x, x^3 e^x \) linearly independent or dependent on \((-\infty, \infty)\)?

Problem 9
Find the largest interval \((a, b)\) on which the following differential equation has a unique solution:

\[(x - 2)y'' + \ln(x)y' = \sqrt{3 - x}\]

with \(y(1) = 0, y'(1) = 2\).
Problem 10

(a) Solve $y''' - 3y'' + 3y' - y = 0$

(b) Find the form of a particular solution to $y''' - 3y'' + 3y' - y = e^t$

Problem 11

Solve $y'' + y = \tan(t)$ using variation of parameters.