1 Diagonalization

Problem 1
Find a diagonal matrix $D$ and an invertible matrix $P$ such that $A = PDP^{-1}$, where:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

2 Orthogonality

Problem 2
Apply the Gram-Schmidt process to find an orthonormal basis for $W = \text{Span} \{u_1, u_2, u_3\}$, where:

$$u_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$$

Problem 3
Find the orthogonal projection of $f(x) = \cos(x)$ on $W$, where:

$W = \text{Span} \{\sin(x), \sin(2x), \cos(2x)\}$

with respect to the following inner product:

$$f \cdot g = \int_{-\pi}^{\pi} f(x)g(x)dx$$
3 Symmetric matrices

Problem 4
Find an orthogonal matrix $P$ and a diagonal matrix $D$ such that $A = PDP^T$, where:

$$A = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$$

Problem 5
Write the quadratic form $x_1^2 - 6x_1x_2 + 9x_2^2$ without cross-product terms.

4 Vector Spaces

Problem 6
Let $\mathcal{B} = \{e^x, e^x \cos(x), e^x \sin(x)\}$, and let $V = \text{Span}(\mathcal{B})$.

Define $T : V \rightarrow V$ by:

$$T(y) = y' + 2y$$

Find the matrix of $T$ relative to $\mathcal{B}$.

Problem 7
Let $V$ be the vector space of $2 \times 2$ symmetric matrices. Find a basis for $V$ and the dimension of $V$.

Problem 8
For the following matrix $A$, find $\text{Rank}(A)$ and a basis for $\text{Row}(A), \text{Col}(A), \text{Nul}(A)$:

$$A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
5 True/False Extravaganza!

Problem 9
(a) If $\text{Nul}(A) = \{0\}$, then $\text{Rank}(A)$ is the number of columns of $A$.

(b) If $A$ is a $6 \times 8$ matrix, then the smallest possible dimension of $\text{Nul}(A)$ is 6.

(c) If $\dim(V) = 3$ and $T : V \to V$ is one-to-one, then it is also onto.

(d) If $Q$ is an $n \times n$ orthogonal matrix, then $\det(Q) = \pm 1$.

(e) If $A$ is symmetric, then eigenvectors corresponding to different eigenvalues are orthogonal.

(f) If $W$ is a subspace of $V$ and $y \in V$, then there is a unique vector $\tilde{w}$ in $W$ such that $\|y - \tilde{w}\| \leq \|y - w\|$ for all $w \in W$.

(g) If $A$ diagonalizable, then so is $A^2$.

(h) If the characteristic polynomial of $A$ is $(\lambda - 1)^3$, then $A$ has 3 linearly independent eigenvectors.

(i) If $A$ is an orthogonal $n \times n$ matrix, then $\text{Row}(A) = \text{Col}(A)$.

(j) Linear algebra is so much more awesome than differential equations! :)

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