Name: ________________________________

Instructions: This is a mock midterm and it’s designed to give you an idea of what the actual midterm will look like.

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Date: Friday, July 13th, 2012.
1. (10 points, 2 points each)

Label the following statements as T or F.

**NOTE:** In this question, you do NOT have to show your work! Don’t spend too much time on each question!

(a) If $\dim(V) = 3$ and $u$ and $v$ are two vectors in $V$, then $\{u, v\}$ cannot be linearly independent!

(b) If $T$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, and $T$ is onto, then $T$ is also one-to-one.

(c) If $A$ is a $m \times n$ matrix, then $Col(A)$ is a subspace of $\mathbb{R}^n$.

(d) If $P^C \leftarrow B$ is the change-of-coordinates matrix from $B = \{b_1, b_2\}$ to $C = \{c_1, c_2\}$ then $P^C \leftarrow B = [c_1|_B, c_2|_B]$

(e) The Span of any set of vectors is always a vector space.
2. (20 points, 5 points each) Label the following statements as TRUE or FALSE. In this question, you HAVE to justify your answer!!

This means:

- If the answer is TRUE, you have to explain WHY it is true (possibly by citing a theorem)

- If the answer is FALSE, you have to give a specific COUNTEREXAMPLE. You also have to explain why the counterexample is in fact a counterexample to the statement!

(a) The set of $2 \times 2$ matrices such that $\det(A) = 0$ is a vector space.

(b) A $4 \times 5$ matrix $A$ cannot be invertible

**Hint:** How big is $\text{Nul}(A)$?
(c) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, the set of $2 \times 2$ matrices $B$ such that $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a vector space.

(d) The set $\{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\}$ is a basis for $P_2$. 
3. (5 points) Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which first reflects points in $\mathbb{R}^2$ about the line $y = x$ and then rotates them by 180 degrees ($\pi$ radians) counterclockwise.
4. (5 points) A $2 \times 2$ matrix is called \textbf{symmetric} if $A^T = A$. Find a basis for the vector space $V$ of all $2 \times 2$ symmetric matrices. Show that the basis you found is in fact a basis!

\textbf{Hint:} What does a general $2 \times 2$ symmetric matrix look like?
5. (10 points) For the following matrix \( A \), find a basis for \( \text{Nul}(A) \), \( \text{Row}(A) \), \( \text{Col}(A) \), and find \( \text{Rank}(A) \):

\[
A = \begin{bmatrix}
3 & -1 & 7 & 3 & 9 \\
-2 & 2 & -2 & 7 & 6 \\
-5 & 9 & 3 & 3 & 4 \\
-2 & 6 & 6 & 3 & 7
\end{bmatrix}
\sim
\begin{bmatrix}
3 & -1 & 7 & 3 & 9 \\
0 & 2 & 4 & 0 & 3 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
6. (10 points) Let $B = \left\{ \begin{pmatrix} -1 \\ 8 \\ \end{pmatrix}, \begin{pmatrix} 1 \\ -5 \\ \end{pmatrix} \right\}$, and $C = \left\{ \begin{pmatrix} 1/4 \\ 1 \\ \end{pmatrix} \right\}$ be bases for $\mathbb{R}^2$.

(a) Find the change-of-coordinates matrix from $B$ to $C$, namely:

$$ P^C \leftarrow B $$

(b) Calculate $[x]_C$ given $[x]_B = \begin{pmatrix} 2 \\ 3 \\ \end{pmatrix}$. 
7. (10 points) Let $V = \text{Span} \{ e^x, e^x \cos(x), e^x \sin(x) \}$, and define $T : V \to V$ by:

$$T(y) = y' + y$$

(a) Show $T$ is linear

(b) Find the matrix of $T$ with respect to the basis $B = \{ e^x, e^x \cos(x), e^x \sin(x) \}$ for $V$.

Note: Don’t freak out! I know this is a brand new problem, but just do the same think you usually do to find matrices of linear transformations!
8. (5 points) Find the largest interval \((a, b)\) on which the following differential equation has a unique solution:

\[
\sin(x) y'' + (\sqrt{2-x}) y' = e^x
\]

with

\[
y\left(\frac{\pi}{2}\right) = 4, \quad y'\left(\frac{\pi}{2}\right) = 0
\]
9. (10 points) Solve the following differential equation:

\[ y''' - 12y'' + 41y' - 42y = 0 \]

**Hint:** 42 = 2 × 3 × 7
10. (10 points)
(a) Solve $y'' + 4y' + 4y = e^{3t}$ using undetermined coefficients.

(b) Solve $y'' + y = \tan(t)$ using variation of parameters.

**Note:** You may need to use the fact that $\tan(t) = \frac{\sin(t)}{\cos(t)}$. Also, you may use the fact that $\int \frac{\sin^2(t)}{\cos(t)}\,dt = \ln \left| \frac{\cos(t)}{\sin(t)-1} \right| - \sin(t)$.
11. (5 points) Suppose $u, v, w$ are linearly dependent vectors (in $V$) and $T : V \to W$ is a linear transformation. Show that $T(u)$, $T(v)$, $T(w)$ are also linearly dependent.

**Hint:** Write down what it means for 3 vectors to be linearly dependent!