1. (10 points, 2 points each)

Label the following statements as T or F.

(a) **FALSE** If $\dim(V) = 3$ and $u$ and $v$ are two vectors in $V$, then $\{u, v\}$ cannot be linearly independent!

(They *could* be linearly independent. For example, take $V = \mathbb{R}^3$, and $u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$! What *is* true, however, is that they cannot span $V$.)

(b) **TRUE** If $T$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, and $T$ is onto, then $T$ is also one-to-one.

(This is the third miracle of Linear Algebra that I’ve been talking about! If you want to prove it, use the rank-nullity theorem!)

(c) **FALSE** If $A$ is a $m \times n$ matrix, then $Col(A)$ is a subspace of $\mathbb{R}^n$.

(It’s a subspace of $\mathbb{R}^m$. For example, take $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, which is a $2 \times 3$ matrix, then $Col(A) = Span \left\{ \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\}$, which is a subspace of $\mathbb{R}^2$. In general, it’s always good to write down an example of what $A$ looks like, so that you have an idea)
of what’s going on!)

(d) **FALSE** If $\mathcal{C} \leftrightarrow \mathcal{B}$ is the change-of-coordinates matrix from $\mathcal{B} = \{b_1, b_2\}$ to $\mathcal{C} = \{c_1, c_2\}$ then $\mathcal{C} \leftrightarrow \mathcal{B} = \begin{bmatrix} [c_1]_B & [c_2]_B \end{bmatrix}

(It’s $\mathcal{C} \leftrightarrow \mathcal{B} = \begin{bmatrix} [b_1]_C & [b_2]_C \end{bmatrix}$, you always take the old vectors (in $\mathcal{B}$) and evaluate them with respect to the new and cool basis $\mathcal{C}$)

(e) **TRUE** The Span of any set of vectors is always a vector space.

(see example 10 on page 209 for example)

2. (20 points, 5 points each) Label the following statements as **TRUE** or **FALSE**. In this question, you **HAVE** to **justify** your answer!!!

This means:

- If the answer is **TRUE**, you have to explain **WHY** it is true (possibly by citing a theorem)

- If the answer is **FALSE**, you have to give a specific **COUNTEREXAMPLE**. You also have to explain why the counterexample is in fact a counterexample to the statement!

(a) **FALSE** The set $V$ of $2 \times 2$ matrices such that $\text{det}(A) = 0$ is a vector space.

Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\text{det}(A) = 0$ and $\text{det}(B) = 0$ so $A$ and $B$ are in $V$. But $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,
so \( \det(A + B) = 1 \neq 0 \), so \( A + B \) is not in \( V \). Hence \( V \) is not closed under addition, and hence is not a vector space.

(b) \( \text{TRUE} \) A \( 4 \times 5 \) matrix \( A \) cannot be invertible

**Hint:** How big is \( \text{Nul}(A) \)?

By the rank-nullity theorem, \( \text{dim}(\text{Nul}(A)) + \text{rank}(A) = 5 \).
But \( \text{Rank}(A) = \text{number of pivots} \), which is at most 4 (since \( A \) has 4 rows). Hence \( \text{dim}(\text{Nul}(A)) \geq 5 - 4 = 1 \). So \( \text{Nul}(A) \neq \{0\} \), hence \( A \) is not invertible.

(c) \( \text{TRUE} \) If \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \), the set \( V \) of \( 2 \times 2 \) matrices \( B \) such that \( AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) is a vector space.

**Note:** By \( O \), I mean \( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \).

First of all, \( V \) is a subset of \( M_{2 \times 2} \), the vector space of \( 2 \times 2 \) matrices.

1) **Zero-vector:** \( AO = O \), so the \( O \)-matrix is in \( V \)

2) **Closed under addition:** \( B \) and \( C \) are in \( V \), then \( AB = O \), and \( AC = O \), so \( A(B + C) = AB + AC = O + O = O \), so \( B + C \) is in \( V \)

3) **Closed under scalar multiplication:** If \( B \) is in \( V \) and \( c \) is in \( \mathbb{R} \), \( AB = O \), and so \( A(cB) = cAB = c(O) = O \), so \( cB \) in \( V \)

Hence \( V \) is a subspace of \( M_{2 \times 2} \) and hence is a vector space.

(d) \( \text{TRUE} \) The set \( \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\} \) is a basis for \( P_2 \)

First of all, identifying polynomials with a number code, we see that all we need to show is whether:
\[ B = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\} \text{ is a basis for } \mathbb{R}^3 \]

Linear independence: To show that \( B \) is linearly independent, form the matrix \( A = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix} \). All we need to show is that \( Ax = 0 \) implies that \( x = 0 \), where \( x = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \). But if you row-reduce \( A \), then you should get:

\[
\begin{bmatrix} 1 & 3 & 0 & 0 \\ -2 & -5 & 2 & 0 \\ 1 & 4 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\]

Which implies that \( x = 0 \), hence \( B \) is linearly independent.

Span: Since \( \dim(\mathbb{R}^3) = 3 \), and \( B \) is a linearly independent with 3 vectors, we get that \( B \) spans \( \mathbb{R}^3 \) (this is one of the shortcuts I’ve been talking about in class).

Therefore \( B \) is a basis for \( \mathbb{R}^3 \), and hence \( \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\} \) is a basis for \( P_2 \).

3. (5 points) Find the matrix of the linear transformation \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) which first reflects points in \( \mathbb{R}^2 \) about the line \( y = x \) and then rotates them by 180 degrees (\( \pi \) radians) counterclockwise.

We have:

\[
T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}
\]

Hence the matrix of \( T \) is:
4. *(5 points)* A $2 \times 2$ matrix is called **symmetric** if $A^T = A$. Find a basis for the vector space $V$ of all $2 \times 2$ symmetric matrices. Show that the basis you found is in fact a basis!

**Hint:** What does a general $2 \times 2$ symmetric matrix look like?

A general $2 \times 2$ symmetric matrix has the form: $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$.

Notice that:

$$
\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
$$

We claim that:

$$
B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}
$$

is a basis for $V$.

**Span:** We just showed that! Any symmetric matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

**Linear Independence:** (this part is **important**) Suppose:

$$
a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
$$

Then:

$$
\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
$$
And hence \( a = 0, b = 0, c = 0 \), and hence the set is linearly independent!

Therefore \( B \) is a basis for \( V \) (and hence \( V \) is 3-dimensional, but you didn’t have to write this).

5. (10 points) For the following matrix \( A \), find a basis for \( \text{Nul}(A) \), \( \text{Row}(A) \), \( \text{Col}(A) \), and find \( \text{Rank}(A) \):

\[
A = \begin{bmatrix}
  3  & -1 & 7 & 3 & 9 \\
 -2 &  2 &-2 & 7 & 6 \\
-5 &  9 & 3 & 3 & 4 \\
-2 &  6 & 6 & 3 & 7 \\
\end{bmatrix}
\sim \begin{bmatrix}
  3  & -1 & 7 & 3 & 9 \\
   0 &  2 & 4 & 0 & 3 \\
  0 &  0 & 1 & 1 \\
  0 &  0 & 0 & 0 \\
\end{bmatrix}
\]

\( \text{Nul}(A) \) Since the right-hand-side is not in reduced row-echelon form, let’s further row-reduce it:

\[
\begin{bmatrix}
  3  & -1 & 7 & 3 & 9 \\
  0 &  2 & 4 & 0 & 3 \\
  0 &  0 & 1 & 1 \\
  0 &  0 & 0 & 0 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
  3  & -1 & 7 & 0 & 6 \\
  0 &  2 & 4 & 0 & 3 \\
  0 &  0 & 1 & 1 \\
  0 &  0 & 0 & 0 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
  3  & 0 & 9 & 0 & \frac{15}{2} \\
  0 & 2 & 4 & 0 & 3 \\
  0 & 0 & 1 & 1 \\
  0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(I first subtracted 3 times the third row from the first row, and then added \( \frac{1}{2} \) times the second row to the first row)

Now if \( A\mathbf{x} = \mathbf{0} \), where \( \mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ t \\ s \end{bmatrix} \), then we get:

\[
\begin{align*}
3x + 9z + \frac{15}{2}s &= 0 \\
2y + 4z + 3s &= 0 \\
t + s &= 0
\end{align*}
\]

That is:
\[
\begin{align*}
  x &= -3z - \frac{5}{2}s \\
  y &= -2z - \frac{3}{2}s \\
  t &= -s
\end{align*}
\]

Hence we get:

\[
\begin{bmatrix}
  x \\
  y \\
  z \\
  t \\
  s
\end{bmatrix} =
\begin{bmatrix}
  -3z - \frac{5}{2}s \\
  -2z - \frac{3}{2}s \\
  z \\
  -s \\
  s
\end{bmatrix} = z \begin{bmatrix}
  -3 \\
  -2 \\
  1 \\
  0 \\
  0
\end{bmatrix} + s \begin{bmatrix}
  -\frac{5}{2} \\
  -\frac{3}{2} \\
  0 \\
  -1 \\
  1
\end{bmatrix}
\]

And therefore:

\[
Nul(A) = Span \left\{ \begin{bmatrix}
  -3 \\
  -2 \\
  1 \\
  0 \\
  0
\end{bmatrix}, \begin{bmatrix}
  \frac{5}{2} \\
  \frac{3}{2} \\
  0 \\
  -1 \\
  1
\end{bmatrix} \right\}
\]

\textit{Row}(A) \quad \text{Notice that there are pivots in the first, second, and third row, hence:}

\[
Row(A) = Span \left\{ \begin{bmatrix}
  3 \\
  -1 \\
  7 \\
  0 \\
  6
\end{bmatrix}, \begin{bmatrix}
  0 \\
  2 \\
  4 \\
  0 \\
  3
\end{bmatrix}, \begin{bmatrix}
  0 \\
  0 \\
  1 \\
  1 \\
  1
\end{bmatrix} \right\}
\]

\textit{Col}(A) \quad \text{Notice that there are pivots in the first, second, and fourth columns, hence:}

\[
Col(A) = Span \left\{ \begin{bmatrix}
  3 \\
  -2 \\
  -5 \\
  -2
\end{bmatrix}, \begin{bmatrix}
  -1 \\
  2 \\
  9 \\
  6
\end{bmatrix}, \begin{bmatrix}
  3 \\
  7 \\
  3 \\
  3
\end{bmatrix} \right\}
\]

(Notice that you had to go back to the matrix \( A \) to find a basis for \( Col(A) \))

\textit{Rank}(A) \quad \text{There are 3 pivots, hence} \quad \textit{Rank}(A) = 3.
6. (10 points) Let $B = \left\{ \begin{bmatrix} -1 \\ 8 \\ \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \end{bmatrix} \right\}$, and $C = \left\{ \begin{bmatrix} 1 \\ 4 \\ \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ be bases for $\mathbb{R}^2$.

(a) Find the change-of-coordinates matrix from $B$ to $C$, namely: 
$$P^C \leftarrow B = \left[ \begin{array}{ccc} 1 & 1 & -1 \\ 4 & 1 & 8 \end{array} \right]$$

(first I added $-4$ times the second row to the first, then I divided row 2 by $-3$, then I subtracted the second row from the first row)

Hence:
$$P^C \leftarrow B = \left[ \begin{array}{ccc} 3 & -2 \\ -4 & 3 \end{array} \right]$$

(b) Calculate $[x]_C$ given $[x]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

We have:
$$[x]_C = C \leftarrow B [x]_B = \left[ \begin{array}{ccc} 3 & -2 \\ -4 & 3 \end{array} \right] \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

7. (10 points) Let $V = \text{Span} \left\{ e^x, e^x \cos(x), e^x \sin(x) \right\}$, and define $T : V \rightarrow V$ by:

$$T(y) = y' + y$$

(a) Show $T$ is linear

$$T(y_1+y_2) = (y_1+y_2)'+(y_1+y_2) = (y_1)'+(y_2)'+(y_1+y_2) = (y_1)'+y_1+(y_2)'+y_2 = T(y_1)+T(y_2)$$
\[ T(cy) = (cy)' + cy = cy' + cy = c(y' + y) = cT(y) \]

Hence \( T \) is a linear transformation.

(b) Find the matrix of \( T \) with respect to the basis \( B = \{ e^x, e^x \cos(x), e^x \sin(x) \} \) for \( V \).

Again, don’t freak out! For every vector/function in \( B \), evaluate \( T \) of that function, and express your answer as a linear combination of the functions in \( B \).

\[
T(e^x) = (e^x)' + e^x \\
= e^x + e^x \\
= 2e^x \\
= 2e^x + 0e^x \cos(x) + 0e^x \sin(x)
\]

\[
T(e^x \cos(x)) = (e^x \cos(x))' + e^x \cos(x) \\
= e^x \cos(x) - e^x \sin(x) + e^x \cos(x) \\
= 2e^x \cos(x) - e^x \sin(x) \\
= 0e^x + 2e^x \cos(x) + (-1)e^x \sin(x)
\]

\[
T(e^x \sin(x)) = (e^x \sin(x))' + e^x \sin(x) \\
= e^x \sin(x) + e^x \cos(x) + e^x \sin(x) \\
= e^x \cos(x) + 2e^x \sin(x) \\
= 0e^x + 1e^x \cos(x) + 2e^x \sin(x)
\]

Hence the matrix of \( T \) is (just put the numbers in bold together in columns):

\[
A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}
\]
8. (5 points) Find the largest interval \((a, b)\) on which the following differential equation has a unique solution:

\[
\sin(x)y'' + \left(\sqrt{2-x}\right)y' = e^x
\]

with

\[
y\left(\frac{\pi}{2}\right) = 4, \ y'\left(\frac{\pi}{2}\right) = 0
\]

First convert the equation in standard form:

\[
y'' + \left(\frac{\sqrt{2-x}}{\sin(x)}\right)y' = \frac{e^x}{\sin(x)}
\]

Now let’s look at the domain of each term:

The domain of \(\frac{\sqrt{2-x}}{\sin(x)}\) is \((-\infty, 2] \cap \{x \neq n\pi\}\) (Basically \((-\infty, 2]\) without multiples of \(\pi\), \(\cap\) means ‘intersection’). The part of that domain which contains the initial condition \(\frac{\pi}{2}\) is \((0, 2]\).

The domain of \(\frac{e^x}{\sin(x)}\) is \(\{x \neq n\pi\}\) (anything except multiples of \(\pi\)). The part of that domain which contains the initial condition \(\frac{\pi}{2}\) is \((0, \pi]\).

And if you intersect the two domains you found you get that the answer is \((0, 2]\).

Note: Make sure your answer is always an open interval! For example, here we got \((0, 2]\), but since it is not an open interval, we chose \((0, 2)\).

9. (10 points) Solve the following differential equation:

\[
y''' - 12y'' + 41y' - 42y = 0
\]

Hint: \(42 = 2 \times 3 \times 7\).
The auxiliary equation is \( r^3 - 12r^2 + 41r - 42 = 0 \).

Now, by the rational roots theorem, we know that if the above polynomial has a rational root, then \( r = \frac{a}{b} \), where \( a \) divides the constant term \(-42\) and \( b \) divides the leading term 1.

The only integers which divide \(-42\) are (by the hint): \( \pm 1, \pm 2, \pm 3, \pm 7, \pm 6, \pm 14, \pm 21, \pm 42 \).

And the only integers which divide 1 are \( \pm 1 \). Hence our guesses are: \( \pm 1, \pm 2, \pm 3, \pm 7, \pm 6, \pm 14, \pm 21, \pm 42 \).

If you plug-and-chug, you eventually figure out that \( r = 2 \) works, i.e. \( r = 2 \) is a root of the auxiliary polynomial.

Now all you have to do is use long division and divide \( r^3 - 12r^2 + 41r - 42 \) by \( r - 2 \):

\[
\begin{align*}
X - 2 & \quad \overline{X^2 - 10X + 21} \\
X^3 - 12X^2 + 41X - 42 & \\
- X^3 + 2X^2 & \\
\quad \underline{- 10X^2 + 41X} \\
10X^2 - 20X & \\
\quad \underline{21X - 42} \\
\quad \underline{- 21X + 42} \\
0 & \\
\end{align*}
\]

In other words, \( r^3 - 12r^2 + 41r - 42 = (r^2 - 10r + 21)(r - 2) = (r - 3)(r - 7)(r - 2) = (r - 2)(r - 3)(r - 7) \).

It follows that the roots are: \( r = 2, 3, 7 \), which means that the general solution is:

\[
y(t) = Ae^{2t} + Be^{3t} + Ce^{7t}
\]

**NOTE:** There are **NO** shortcuts to this problem! In particular, if you don’t show the long division step on the exam, you will lose points!!!
10. (10 points)
(a) Solve $y'' + 4y' + 4y = e^{3t}$ using undetermined coefficients

Homogeneous solution: $r^2 + 4r + 4 = (r + 2)^2 = 0$, which gives $r = -2$, a double root, hence $y_0(t) = Ae^{-2t} + Bte^{-2t}$.

Particular solution: Try $y_p = Ae^{3t}$. If you plug this into the differential equation, you get:

$9Ae^{3t} + 12Ae^{3t} + 4Ae^{3t} = e^{3t} \Rightarrow 25A = 1 \Rightarrow A = \frac{1}{25}$

So a particular solution is: $y_p(t) = \frac{1}{25}e^{3t}$

General solution:

$y(t) = y_0(t) + y_p(t) = Ae^{-2t} + Bte^{-2t} + \frac{1}{25}e^{3t}$

(b) Solve $y'' + y = \tan(t)$ using variation of parameters

Note: You may need to use the fact that $\tan(t) = \frac{\sin(t)}{\cos(t)}$. Also you may use the fact that $\int \frac{\sin^2(t)}{\cos(t)} \, dt = \ln \left| \frac{\cos(t)}{\sin(t) - 1} \right| - \sin(t)$.

Homogeneous solution: $r^2 + 1 = 0$, so $r = \pm i$, so $y_0(t) = A\cos(t) + B\sin(t)$

Particular solution: The Wronskian matrix is $\widetilde{W}(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$.

Now $y_p(t) = v_1(t)\cos(t) + v_2(t)\sin(t)$, where $v_1$ and $v_2$ solve:

$\widetilde{W}(t) \begin{bmatrix} v'_1(t) \\ v'_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \tan(t) \end{bmatrix}$

That is:

$\begin{bmatrix} v'_1(t) \\ v'_2(t) \end{bmatrix} = \left( \widetilde{W}(t) \right)^{-1} \begin{bmatrix} 0 \\ \tan(t) \end{bmatrix}$

But:
\((\tilde{W}(t))^{-1} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}^{-1} = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}\)

Hence:

\[
\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 0 \\ \tan(t) \end{bmatrix}
\]

Therefore:

\[
\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} -\sin(t) \tan(t) \\ \cos(t) \tan(t) \end{bmatrix}
\]

That is:

\[
v_1'(t) = -\sin(t) \tan(t) = -\sin(t) \left( \frac{\sin(t)}{\cos(t)} \right) = -\frac{\sin^2(t)}{\cos(t)}
\]

So:

\[
v_1(t) = -\int \frac{\sin^2(t)}{\cos(t)} = -\ln \left| \frac{\cos(t)}{\sin(t) - 1} \right| + \sin(t)
\]

(by the Hint, and ignore the constant)

And

\[
v_2'(t) = \cos(t) \tan(t) = \cos(t) \left( \frac{\sin(t)}{\cos(t)} \right) = \sin(t)
\]

So:

\[
v_2(t) = -\cos(t)
\]

Therefore:

\[
y_p(t) = v_1(t) \cos(t) + v_2(t) \sin(t)
\]

\[
= \left( -\ln \left| \frac{\cos(t)}{\sin(t) - 1} \right| + \sin(t) \right) \cos(t) - \cos(t) \sin(t)
\]

\[
= -\ln \left| \frac{\cos(t)}{\sin(t) - 1} \right| \cos(t)
\]

General solution:

\[
y(t) = y_0(t) + y_p(t) = A \cos(t) + B \sin(t) - \ln \left| \frac{\cos(t)}{\sin(t) - 1} \right| \cos(t)
\]
11. (5 points) Suppose \( u, v, w \) are linearly dependent vectors (in \( V \)) and \( T : V \to W \) is a linear transformation. Show that \( T(u), T(v), T(w) \) are also linearly dependent.

**Hint:** Write down what it means for 3 vectors to be linearly dependent!

We know that for \( a, b, c \not\equiv 0 \):

\[
a u + b v + c w = 0
\]

Now apply \( T \) to this equation:

\[
T(a u + b v + c w) = T(0) = 0
\]

But since \( T \) is linear, we have:

\[
a T(u) + b T(v) + c T(w) = 0
\]

But since \( a, b, c \) are not all \( 0 \), this also means that the vectors \( T(u), T(v), T(w) \) are linearly dependent! And we’re done!