MATH 54 – MIDTERM 3

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Name: ________________________________

**Instructions:** This midterm counts for 20% of your grade. You officially have 90 minutes to take this exam. May your luck be diagonalizable! :)

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1. (10 points, 2 points each)

Label the following statements as T or F. **Write your answers in the box below!**

**NOTE:** In this question, you do NOT have to show your work! Don’t spend *too* much time on each question!

(a) If $A$ is diagonalizable, then $A^3$ is diagonalizable.

(b) If $A$ is a $3 \times 3$ matrix with 3 (linearly independent) eigenvectors, then $A$ is diagonalizable.

(c) If $A$ is a $3 \times 3$ matrix with eigenvalues $\lambda = 1, 2, 3$, then $A$ is invertible.

(d) If $A$ is a $3 \times 3$ matrix with eigenvalues $\lambda = 1, 2, 3$, then $A$ is (always) diagonalizable.

(e) If $A$ is a $3 \times 3$ matrix with eigenvalues $\lambda = 1, 2, 2$, then $A$ is (always) not diagonalizable.

\begin{array}{|c|}
\hline
(a) \\
(b) \\
(c) \\
(d) \\
(e) \\
\hline
\end{array}
2. *(15 points)* Label the following statements as **TRUE** or **FALSE**. In this question, you **HAVE** to justify your answer!!!

This means:

- If the answer is **TRUE**, you have to explain **WHY** it is true (possibly by citing a theorem)

- If the answer is **FALSE**, you have to give a specific **COUNTEREXAMPLE**. You also have to explain why the counterexample is in fact a counterexample to the statement!

**IMPORTANT NOTE:** If $A$ is diagonalizable, explain why! And if $A$ is not diagonalizable, *show* why it isn’t!

(a) *(5 points)* If $A$ is diagonalizable, then $A$ is invertible.
(b) (10 points, longer) If $A$ is invertible, then $A$ is diagonalizable
NO MORE JUSTIFICATION T/F

IN YOUR LIFE!!!
3. (30 points) Find a diagonal matrix $D$ and an invertible matrix $P$ such that $A = PDP^{-1}$, where:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

**Note:** Show all your work!

**Helps:** In case you’re stuck:

- For 5 points, I can give you one root of the characteristic polynomial (ask me about it)
- For 10 points, I can give you the full characteristic polynomial! (ask me about it)
(Continuation)
4. (25 points) Solve the following system $\mathbf{x}' = A\mathbf{x}$, where:

$$A = \begin{bmatrix}
0 & 5 & 0 \\
-1 & 4 & 0 \\
0 & 0 & 2
\end{bmatrix}$$

Note: Show all your work!
(Continuation)
5. (20 points, 10 points each)

Find the general solution to $x' = Ax + f$, where:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, f(t) = \begin{bmatrix} e^{4t} \\ e^{4t} \end{bmatrix}$$

**Note:** You may use the fact that the general solution to $x' = Ax$ is:

$$x_0(t) = Ae^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + Be^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(a) (10 points) Using undetermined coefficients
(b) (10 points) Using variation of parameters

**Note:** Simplify your answer!
Bonus (2 points)

(a) (0 points, but it’ll help you for (b)) What is the general solution of \( y'' = -b^2 y \)

(b) (2 points) Use (a) and the ideas we talked about in lecture about the matrix exponential function to solve the following system \( x'' = Ax \) (note the double prime), where:

\[
A = \begin{bmatrix} 2 & -3 \\ 6 & 7 \end{bmatrix}
\]

**Hint:** You may use the fact that \( A = -B^2 \), where \( B = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \)
as well as the fact that \( B = PDP^{-1} \), where \( P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \),

\[
D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}
\]

**Note:** Converting this into a first-order system differential equations is a complete waste of time! Do it directly using the hint above!

**Note:** This problem is a tiny bit long, but you really need to write down the final answer to get full credit!
(Continuation)
(Scratch work)
Any comments about this exam? (too long? too hard?)