Section 5.1 (Page 241)

5. Is \[
\begin{bmatrix}
3 \\
-2 \\
1
\end{bmatrix}
\] an eigenvector for \[
\begin{bmatrix}
-4 & 3 & 3 \\
2 & -3 & -2 \\
-1 & 0 & -2
\end{bmatrix}
\]? If so, find the eigenvalue.

Multiplying them gives
\[
\begin{bmatrix}
-4 & 3 & 3 \\
2 & -3 & -2 \\
-1 & 0 & -2
\end{bmatrix}
\begin{bmatrix}
3 \\
-2 \\
1
\end{bmatrix}
= \begin{bmatrix}
-15 \\
10 \\
-5
\end{bmatrix}
= -5 \begin{bmatrix}
3 \\
-2 \\
1
\end{bmatrix}.
\]

This shows that the vector is an eigenvector for the eigenvalue \(-5\).

12. Find a basis for the eigenspace corresponding to each listed eigenvalue:

\[
A = \begin{bmatrix}
4 & 1 \\
3 & 6
\end{bmatrix}, \quad \lambda = 3, 7
\]

The eigenspace for \(\lambda = 3\) is the null space of \(A - 3I\), which is row reduced as follows:

\[
\begin{bmatrix}
1 & 1 \\
3 & 3
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 \\
0 & 0
\end{bmatrix}.
\]

The solution is \(x_1 = -x_2\) with \(x_2\) free, and the basis is \(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\).

For \(\lambda = 7\), row reduce \(A - 7I\):

\[
\begin{bmatrix}
-3 & 1 \\
3 & -1
\end{bmatrix} \sim \begin{bmatrix}
-3 & 1 \\
0 & 0
\end{bmatrix}.
\]

The solution is \(3x_1 = x_2\) with \(x_2\) free, and the basis is \(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\).

So the bases for the eigenspaces for \(\lambda = 3\) and \(\lambda = 7\) are (respectively)

\[
\text{Span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} \quad \text{and} \quad \text{Span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}.
\]

15. Find a basis for the eigenspace corresponding to each listed eigenvalue:

\[
A = \begin{bmatrix}
-4 & 1 & 1 \\
2 & -3 & 2 \\
3 & 3 & -2 \\
1
\end{bmatrix}, \quad \lambda = -5.
\]
Find the null space of $A - \lambda I$ by row reducing it:

\[
\begin{bmatrix}
1 & 1 & 1 \\
2 & 2 & 2 \\
3 & 3 & 3
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

So $x_2$ and $x_3$ are free variables. In parametrized vector form, the null space is

\[
x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},
\]

so a basis for the eigenspace is given by the two vectors above.

25. Let $\lambda$ be an eigenvalue of an invertible matrix $A$. Show that $\lambda^{-1}$ is an eigenvalue of $A^{-1}$. \textit{[Hint: suppose a nonzero $\vec{x}$ satisfies $A\vec{x} = \lambda \vec{x}$.]}

It is noted just below Example 5 that, since $A$ is invertible, $\lambda$ cannot be zero.

As in the hint, if $A\vec{x} = \lambda \vec{x}$, then multiplying both sides on the left by $A^{-1}$, and also by the scalar $\lambda^{-1}$ gives

\[
\lambda^{-1}\vec{x} = A^{-1}\vec{x}.
\]

Therefore $\lambda^{-1}$ is an eigenvalue for $A^{-1}$, since $\vec{x} \neq \vec{0}$.

26. Show that if $A^2$ is the zero matrix, then the only eigenvalue of $A$ is zero.

Let $\lambda$ be an eigenvalue of $A$, and let $\vec{x}$ be a corresponding eigenvector. Since $A^2 = 0$, we have

\[
\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{x} = A^2 \vec{x} = A\lambda \vec{x} = \lambda^2 \vec{x}.
\]

Since $\vec{x} \neq \vec{0}$, this implies $\lambda^2 = 0$ (e.g., by looking at a nonzero coordinate of $\vec{x}$), and therefore the only possible eigenvalue is $\lambda = 0$.

On the other hand, $\lambda = 0$ is an eigenvalue, because if it wasn’t, then $A$ would be invertible, and so would $A^2$ since it’s a product of invertible matrices. But the zero matrix is not invertible, so 0 must be an eigenvalue.

Section 5.2 (Page 249)

17. For the following matrix, list the real eigenvalues, repeated according to their multiplicities.

\[
\begin{bmatrix}
3 & 0 & 0 & 0 & 0 \\
-5 & 1 & 0 & 0 & 0 \\
3 & 8 & 0 & 0 & 0 \\
0 & -7 & 2 & 1 & 0 \\
-4 & 1 & 9 & -2 & 3
\end{bmatrix}
\]

The matrix is lower triangular, so its eigenvalues are the entries along its main diagonal: 0, 1 (twice), and 3 (twice).
18. It can be shown that the algebraic multiplicity of an eigenvalue \( \lambda \) is always greater than or equal to the dimension of the eigenspace corresponding to \( \lambda \). Find \( h \) in the matrix \( A \) below such that the eigenspace for \( \lambda = 4 \) is two-dimensional.

\[
A = \begin{bmatrix}
4 & 2 & 3 & 3 \\
0 & 2 & h & 3 \\
0 & 0 & 4 & 14 \\
0 & 0 & 0 & 2
\end{bmatrix}
\]

Let’s perform some row operations on the matrix \( A - \lambda I \):

\[
A - \lambda I = \begin{bmatrix}
0 & 2 & 3 & 3 \\
0 & -2 & h & 3 \\
0 & 0 & 14 & 0 \\
0 & 0 & 0 & -2
\end{bmatrix} \sim \begin{bmatrix}
0 & 2 & 3 & 3 \\
0 & 0 & h + 3 & 6 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix} \sim \begin{bmatrix}
0 & 2 & 3 & 3 \\
0 & 0 & h + 3 & 6 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

If \( h \neq -3 \), then the matrix is in echelon form with \( x_1 \) as its only free variable. In that case the eigenspace for \( \lambda = 4 \) will be only one-dimensional. If \( h = -3 \), however, then it is not in echelon form, but only one elementary row operation is needed to put it into echelon form. For that matrix, both \( x_1 \) and \( x_3 \) are free variables, so the eigenspace in question is two-dimensional.

20. Use a property of determinants to show that \( A \) and \( A^T \) have the same characteristic polynomial.

Since \( I = I^T \), the characteristic polynomial of \( A^T \) is:

\[
det(A^T - \lambda I) = det(A^T - \lambda I^T) = det(A^T - (\lambda I)^T) = det((A - \lambda I)^T).
\]

This equals the characteristic polynomial \( det(A - \lambda I) \) of \( A \) since the determinant of the transpose of a matrix is the same as the determinant of the original matrix.

Section 5.3 (Page 256)

24. \( A \) is a \( 3 \times 3 \) matrix with two eigenvalues. Each eigenspace is one-dimensional. Is \( A \) diagonalizable? Why?

No. The sum of the dimensions of the eigenspaces is 2, but it would have to be 3 for the matrix to be diagonalizable (Theorem 7b on page 255).

32. Construct a nondiagonal \( 2 \times 2 \) matrix that is diagonalizable but not invertible.

There are many ways to do this. Certainly 0 has to be an eigenvalue. One way would be to have another eigenvalue, say 1, so we want a matrix whose characteristic polynomial is \( \lambda^2 - \lambda \). If \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), then

\[
\begin{vmatrix}
a - \lambda & b \\
c & d - \lambda
\end{vmatrix} = (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + ad - bc,
\]
so we want \( ad - bc = 0 \) and \( a + d = 1 \). Taking \( a = 1 \) and \( d = 0 \) would lead to \( bc = 0 \), so we can take \( b = 0 \) and (necessarily) \( c \neq 0 \):

\[
\begin{bmatrix}
1 & 0 \\
1 & 0
\end{bmatrix}
\]

Or, with a double eigenvalue \( \lambda = 0 \), we can find a matrix whose null space is one-dimensional, such as

\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\]

**Section 5.4 (Page 263)**

20. Verify that if \( A \) is similar to \( B \), then \( A^2 \) is similar to \( B^2 \).

Since \( A \) and \( B \) are similar, there is an invertible matrix \( P \) for which \( A = PBP^{-1} \). Then \( A^2 = AA = (PBP^{-1})(PBP^{-1}) = PB(P^{-1}P)BP^{-1} = PBBP^{-1} = PB^2P^{-1} \). This shows that \( A^2 \) and \( B^2 \) are also similar.

22. Verify that if \( A \) is diagonalizable and \( B \) is similar to \( A \), then \( B \) is also diagonalizable.

If \( A \) is diagonalizable, then we can write \( A = PDP^{-1} \) with \( P \) invertible and \( D \) diagonal. If \( A \) is similar to \( B \) then we have \( A = QBQ^{-1} \) for some invertible matrix \( Q \). Setting these two expressions for \( A \) equal, we have \( QBQ^{-1} = PDP^{-1} \), so multiplying on the left by \( Q^{-1} \) and on the right by \( Q \) gives

\[
B = Q^{-1}PDP^{-1}Q = (Q^{-1}P)D(Q^{-1}P)^{-1}
\]

and therefore \( B \) and \( D \) are similar since \( Q^{-1}P \) is invertible. In other words, \( B \) is diagonalizable.

25. The *trace* of a square matrix \( A \) is the sum of the diagonal entries in \( A \) and is denoted by \( \text{tr} A \). It can be verified that \( \text{tr}(FG) = \text{tr}(GF) \) for any two \( n \times n \) matrices \( F \) and \( G \). Show that if \( A \) and \( B \) are similar, then \( \text{tr} A = \text{tr} B \).

Since \( A \) and \( B \) are similar, \( B = P^{-1}AP \). Letting \( F = AP \) and \( G = P^{-1} \), we have \( FG = APP^{-1} = A \) and \( GF = P^{-1}AP = B \). Therefore \( \text{tr}(FG) = \text{tr}(GF) \) gives \( \text{tr} A = \text{tr} B \).