Section 2.1 (Page 102)

8. How many rows does $B$ have if $BC$ is a $5 \times 4$ matrix?

The matrix $C$ has to be a $4 \times p$ matrix, and then $BC$ will be a $5 \times p$ matrix, so it will have 5 rows.

10. Let $A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} -3 & -5 \\ 2 & 1 \end{bmatrix}$. Verify that $AB = AC$ and yet $B \neq C$.

$$AB = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -21 & -21 \\ 7 & 7 \end{bmatrix};$$

$$AC = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -3 & -5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -21 & -21 \\ 7 & 7 \end{bmatrix} = AB.$$  

Yet, clearly, $B \neq C$.

23. Suppose $CA = I_n$ (the $n \times n$ identity matrix). Show that the equation $A\vec{x} = \vec{0}$ has only the trivial solution. Explain why $A$ cannot have more columns than rows.

If $A\vec{x} = \vec{0}$, then multiplying both sides on the left by $C$ gives $CA\vec{x} = C\vec{0}$. Since

$$CA\vec{x} = I_n\vec{x} = \vec{x} \quad \text{and} \quad C\vec{0} = \vec{0},$$

this gives $\vec{x} = \vec{0}$, so $\vec{0}$ is the only possible solution to this equation.

The matrix $A$ cannot have more columns than rows, because (by Theorem 8 on page 61), the columns would then be linearly dependent. Then, by the highlighted statement on page 59, $A\vec{x} = \vec{0}$ would have a nontrivial solution, contrary to what was just shown.

34. Give a formula for $(AB\vec{x})^T$, where $\vec{x}$ is a vector and $A$ and $B$ are matrices of appropriate sizes.

By repeated application of Theorem 3(d),

$$(AB\vec{x})^T = (A(B\vec{x}))^T = (B\vec{x})^T A^T = \vec{x}^T B^T A^T.$$
7. Let \(A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}, \ b_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \ b_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \ b_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \text{ and } b_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.\)

(a). Find \(A^{-1}\), and use it to solve the four equations
\[A\vec{x} = b_1, \quad A\vec{x} = b_2, \quad A\vec{x} = b_3, \quad A\vec{x} = b_4.\]

(b). The four equations in part (a) can be solved by the same set of row operations, since the coefficient matrix is the same in each case. Solve the four equations in part (a) by row reducing the augmented matrix
\[
\begin{bmatrix}
A & b_1 & b_2 & b_3 & b_4
\end{bmatrix}.
\]

a. By Theorem 4, \(A^{-1} = \begin{bmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{bmatrix}\). Therefore the solutions are
\[
A^{-1}b_1 = \begin{bmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \end{bmatrix},
\]
\[
\begin{bmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 13 \\ -5 \end{bmatrix}.
\]

b. Row reduction gives
\[
\begin{bmatrix}
A & b_1 & b_2 & b_3 & b_4
\end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 5 & 12 & 3 & -5 & 6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 0 & 2 & 8 & -10 & -4 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -9 & 11 & 6 & 13 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{bmatrix}.
\]

Now read the solutions from the last four columns of the row-reduced matrix.

8. Suppose \(P\) is invertible and \(A = PBP^{-1}\). Solve for \(B\) in terms of \(A\).

Multiply both sides of the equation on the left by \(P^{-1}\) and the right by \(P\), and simplify the right-hand side:
\[P^{-1}AP = P^{-1}(PBP^{-1})P = (P^{-1}P)B(P^{-1}P) = IBI = B.\]
Therefore \(B = P^{-1}AP\).

12. Use matrix algebra to show that if \(A\) is invertible and \(D\) satisfies \(AD = I\), then \(D = A^{-1}\).

Multiplying both sides of the equation on the left by \(A^{-1}\) and simplifying both sides gives
\[A^{-1}(AD) = A^{-1}I; \quad (A^{-1}A)D = A^{-1}; \quad ID = A^{-1}; \quad D = A^{-1}.\]

(Note that we can't use the Invertible Matrix Theorem yet, because that theorem is not introduced until Section 2.3. Also, this exercise is used in the proof of the IMT, so using the IMT would be circular reasoning.)
24. Suppose $A$ is $n \times n$ and the equation $A\vec{x} = \vec{b}$ has a solution for each $\vec{b}$ in $\mathbb{R}^n$. Explain why $A$ must be invertible. [Hint: Is $A$ row equivalent to $I_n$?]

Suppose that $A\vec{x} = \vec{b}$ has a solution for each $\vec{b}$. Then, by Theorem 4 on page 39, $A$ has a pivot position in every row. Therefore $A$ has $n$ pivot positions, so every column of $A$ is a pivot column. It follows that the pivot positions of $A$ are all along the main diagonal. Therefore, in the matrix in reduced echelon form that is (row) equivalent to $A$, all entries along the main diagonal are 1, and all other entries are 0. This matrix must be $I_n$, so $A$ is row equivalent to $I_n$. By Theorem 7 on page 109, it follows that $A$ is invertible.

(Again, we can’t use the IMT here, for the same reason as for Exercise 12.)

38. Let $A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$. Construct a $4 \times 2$ matrix $D$ using only 1 and 0 as entries, such that $AD = I_2$. Is it possible that $CA = I_4$ for some $4 \times 2$ matrix $C$? Why or why not?

There are many possible matrices $D$, including:

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$.

(For a full list, combine any column appearing as the first column of any of the above matrixes, with a column appearing as the second column of any of the above.)

For the second question, assume that $C$ is such a matrix. Then

$$CA = \begin{bmatrix} C & 1 \\ C & 0 \\ C & -1 \\ C & 1 \end{bmatrix}.\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Since this (supposedly) equals $I_4$, we have

$$C\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad C\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

In other words, the first column of $C$ must be $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, and the second must be $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Therefore $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$. However, one can quickly check that $CA \neq I_4$, so no such matrix $C$ exists.

Another way to see this is to use Exercise 23 on page 103.
Section 2.3 (Page 117)

8. Determine whether the matrix
\[
\begin{bmatrix}
3 & 0 & -3 \\
2 & 0 & 4 \\
-4 & 0 & 7
\end{bmatrix}
\]
is invertible. Use as few calculations as possible. Justify your answer.

Not invertible. Expanding along the middle column gives that the determinant is zero.

27. Let $A$ and $B$ be $n \times n$ matrices. Show that if $AB$ is invertible, so is $A$. You cannot use Theorem 6(b), because you cannot assume that $A$ and $B$ are invertible. [Hint: There is a matrix $W$ such that $ABW = I$. Why?]

Let $W = (AB)^{-1}$. Then $ABW = I$, so condition (k) of the IMT is true with $D = BW$. Therefore $A$ is invertible.

36. Suppose a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ has the property that $T(\vec{u}) = T(\vec{v})$ for some pair of distinct vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$. Can $T$ map $\mathbb{R}^n$ onto $\mathbb{R}^n$? Why or why not?

Let $A$ be the standard matrix of $T$. Then the given property says that $\vec{x} \mapsto A\vec{x}$ is not one-to-one. By the IMT (properties (f) and (i)), $T$ is not onto.

Section 3.1 (Page 155)

4. Compute the determinant
\[
\begin{vmatrix}
1 & 3 & 5 \\
2 & 1 & 1 \\
3 & 4 & 2
\end{vmatrix}
\]
using a cofactor expansion across the first row. Also compute the determinant by a cofactor expansion down the second column.

Using the first row:
\[
\begin{vmatrix}
1 & 3 & 5 \\
2 & 1 & 1 \\
3 & 4 & 2
\end{vmatrix} =
\begin{vmatrix}
1 & 1 \\
4 & 2 \\
3 & 2
\end{vmatrix} -
\begin{vmatrix}
2 & 1 \\
3 & 2 \\
4 & 3
\end{vmatrix} +
\begin{vmatrix}
1 & 5 \\
3 & 4 \\
2 & 1
\end{vmatrix}
= (2 - 4) - 3(4 - 3) + 5(8 - 3) = -2 - 3 + 25 = 20 .
\]

Using the second column:
\[
\begin{vmatrix}
1 & 3 & 5 \\
2 & 1 & 1 \\
3 & 4 & 2
\end{vmatrix} =
\begin{vmatrix}
2 & 1 \\
3 & 2 \\
4 & 3
\end{vmatrix} +
\begin{vmatrix}
1 & 5 \\
3 & 2 \\
2 & 1
\end{vmatrix} -
\begin{vmatrix}
2 & 1 \\
3 & 2 \\
4 & 3
\end{vmatrix}
= -3(4 - 3) + (2 - 15) - 4(1 - 10) = -3 - 13 + 36 = 20 .
\]
9. Compute the determinant
\[
\begin{vmatrix}
6 & 0 & 0 & 5 \\
1 & 7 & 2 & -5 \\
2 & 0 & 0 & 0 \\
8 & 3 & 1 & 8 \\
\end{vmatrix}
\]
by cofactor expansions. At each step, choose a row or column that involves the least amount of computation.

Expand along the third row, and then along the first row.
\[
\begin{vmatrix}
6 & 0 & 0 & 5 \\
1 & 7 & 2 & -5 \\
2 & 0 & 0 & 0 \\
8 & 3 & 1 & 8 \\
\end{vmatrix} = 2
\begin{vmatrix}
0 & 0 & 5 \\
7 & 2 & -5 \\
3 & 1 & 8 \\
\end{vmatrix} = 2 \cdot 5
\begin{vmatrix}
7 & 2 \\
3 & 1 \\
\end{vmatrix} = 10(7 - 6) = 10.
\]

12. Compute the determinant
\[
\begin{vmatrix}
4 & 0 & 0 & 0 \\
7 & -1 & 0 & 0 \\
2 & 6 & 3 & 0 \\
5 & -8 & 4 & -3 \\
\end{vmatrix}
\]
by cofactor expansions. At each step, choose a row or column that involves the least amount of computation.

Keep expanding along the first row.
\[
\begin{vmatrix}
4 & 0 & 0 & 0 \\
7 & -1 & 0 & 0 \\
2 & 6 & 3 & 0 \\
5 & -8 & 4 & -3 \\
\end{vmatrix} = 4
\begin{vmatrix}
-1 & 0 & 0 \\
6 & 3 & 0 \\
-8 & 4 & -3 \\
\end{vmatrix} = 4(-1)
\begin{vmatrix}
3 & 0 \\
4 & -3 \\
\end{vmatrix} = 4(-1)(3)(-3) = 36.
\]

17. Use the method for computing $3 \times 3$ determinants described in the book to compute
\[
\begin{vmatrix}
2 & -4 & 3 \\
3 & 1 & 2 \\
1 & 4 & -1 \\
\end{vmatrix}
\]
\[
\begin{vmatrix}
2 & -4 & 3 \\
3 & 1 & 2 \\
1 & 4 & -1 \\
\end{vmatrix} = 2(1)(-1) + (-4)(2)(1) + 3(3)(4) - 1(1)(3) - 4(2)(2) - (-1)(3)(-4)
\]
\[
= -2 - 8 + 36 - 3 - 16 - 12 = -5.
\]

26. Compute the determinant of the elementary matrix
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
k & 0 & 1 \\
\end{bmatrix}
\]
This is a lower triangular matrix, so its determinant is $1 \times 1 \times 1 = 1$. 
38. Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and let \( k \) be a scalar. Find a formula that relates \( \det kA \) to \( k \) and \( \det A \).

\[
\det kA = \begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} = ka(kd) - kb(kc) = k^2(ad - bc) = k^2 \det A.
\]

Section 3.2 (Page 163)

8. Find the determinant

\[
\begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{vmatrix}
\]

by row reduction to echelon form.

\[
\begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & -1 & -2 & 5 \\ 0 & 2 & 4 & -10 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0.
\]

14. Combine the methods of row reduction and cofactor expansion to compute the determinant

\[
\begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -3 & 4 & -2 & 8 \\ 3 & -4 & 0 & 4 \end{vmatrix}.
\]

\[
\begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -3 & 4 & -2 & 8 \\ 3 & -4 & 0 & 4 \end{vmatrix} = \begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -9 & 0 & 0 & 0 \\ 3 & -4 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -3 \\ -9 & 0 & 0 \\ 3 & -4 & 4 \end{vmatrix} = -1(-9) \begin{vmatrix} 3 & -3 \\ -4 & 4 \end{vmatrix} = 9(12 - 12) = 0.
\]

18. Find the determinant

\[
\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix}, \text{ where } \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} = 7.
\]

By repeatedly interchanging rows, we have

\[
\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix} = - \begin{vmatrix} a & b & c \\ g & h & i \end{vmatrix} = + \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} = 7.
\]
21. Use determinants to find out if the matrix \[
\begin{bmatrix}
2 & 3 & 0 \\
1 & 3 & 4 \\
1 & 2 & 1 \\
\end{bmatrix}
\] is invertible.

\[
\begin{vmatrix}
2 & 3 & 0 \\
1 & 3 & 4 \\
1 & 2 & 1 \\
\end{vmatrix} = \begin{vmatrix}
2 & 3 & 0 \\
-3 & -5 & 0 \\
1 & 2 & 1 \\
\end{vmatrix} = 2(3) - 3(5) = 2(-5) - 3(-3) = -1.
\]

Since this is not zero, the matrix is invertible.

25. Use determinants to decide if the set \[
\begin{bmatrix}
7 \\
-4 \\
6 \\
\end{bmatrix}, \begin{bmatrix}
-8 \\
5 \\
7 \\
\end{bmatrix}, \begin{bmatrix}
7 \\
0 \\
-5 \\
\end{bmatrix}
\] is linearly independent.

We have

\[
\begin{vmatrix}
7 & -8 & 7 \\
-4 & 5 & 0 \\
6 & 7 & -5 \\
\end{vmatrix} = 4 \begin{vmatrix}
-8 & 7 \\
7 & -5 \\
\end{vmatrix} + 5 \begin{vmatrix}
7 & 7 \\
6 & -5 \\
\end{vmatrix} = 4(40 - 49) + 5(-35 - 42) = -421.
\]

Since the determinant is nonzero, the matrix is nonsingular, so its columns are linearly independent.

34. Let \(A\) and \(P\) be square matrices, with \(P\) invertible. Show that \(\det(PAP^{-1}) = \det A\).
Mention an appropriate theorem in your explanation.

We have

\[
\det(PAP^{-1}) = (\det(PA))(\det(P^{-1})) = (\det P)(\det A) \cdot \frac{1}{\det P} = \det A.
\]

This is true by Theorem 6 and Exercise 31 (the latter was proved in class on Monday, February 9).