8. Find a formal solution to the initial-boundary value problem.

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0, \]

\[ u(0, t) = 0, \quad u(\pi, t) = 3\pi, \quad t > 0, \]

\[ u(x, 0) = 0, \quad 0 < x < \pi. \]

Much of this problem is similar to Example 2 on pages 604–605. Following this example, we look for a solution of the form

\[ u(x, t) = v(x) + w(x, t) \]

with

\[ \lim_{t \to \infty} w(x, t) = \lim_{t \to \infty} \frac{\partial w}{\partial t}(x, t) = \lim_{t \to \infty} \frac{\partial w}{\partial x}(x, t) = \lim_{t \to \infty} \frac{\partial^2 w}{\partial x^2}(x, t) = 0 \]

(and therefore \( v(x) \) is the steady-state solution and \( w(x, t) \) is the transient part of the solution). (Alternatively, one could think of \( v(x) \) as a particular solution \( u_p \), in the spirit of earlier work with nonhomogeneous ordinary differential equations.)

Substituting this expression for \( u \) into the main differential equation gives

\[ \frac{\partial w}{\partial t}(x, t) = v''(x) + \frac{\partial^2 w}{\partial x^2}(x, t). \]

Letting \( t \to \infty \), the two terms involving \( w \) go to zero, leaving \( v''(x) = 0 \). Twice integrating gives \( v(x) = c_1 x + c_2 \).

Substituting the expression for \( u \) into the boundary conditions gives

\[ v(0) + w(0, t) = 0 \quad \text{and} \quad v(\pi) + w(\pi, t) = 3\pi, \quad t > 0. \]

Again, taking the limit as \( t \to \infty \) causes the \( w \) terms to go away, leaving \( v(0) = 0 \) and \( v(\pi) = 3\pi \). The first of these conditions implies \( c_2 = 0 \), so \( v(x) = c_1 x \). The second of these conditions then gives \( c_1 = 3 \), so \( v(x) = 3x \).

Now consider \( w(x, t) \). Substituting \( u(x, t) = w(x, t) + 3x \) into the differential equations and boundary and initial conditions given in the problem leads to the system of equations

\[ \frac{\partial w}{\partial t}(x, t) = \frac{\partial^2 w}{\partial x^2}(x, t), \quad 0 < x < \pi, \quad t > 0; \]

\[ w(0, t) = w(\pi, t) = 0, \quad t > 0; \]

\[ w(x, 0) = -3x, \quad 0 < x < \pi. \]
(One could also have jumped immediately to this point by citing equations (25)–(27) on page 604.)

This is a problem that can be handled by the method of Exercise 17 on page 600, except that the series for \( f(x) \) is an infinite Fourier series instead of a finite sum. Multiplying the Fourier sine series from Exercise 9 on page 592 by \(-3\) gives

\[
-3x \sim 6 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx,
\]

so \( w(x, t) \) is given as in (4) on page 600 (or by the second term below); and the formal solution to the original problem involving \( u(x, t) \) is

\[
u(x, t) = 3x + 6 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n^2t} \sin nx.
\]

18. Find a formal solution to the initial-boundary value problem.

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad 0 < x < \pi, \quad 0 < y < \pi, \quad t > 0,
\]

\[
\frac{\partial u}{\partial x}(0, y, t) = \frac{\partial u}{\partial x}(\pi, y, t) = 0, \quad 0 < y < \pi, \quad t > 0,
\]

\[
u(x, 0, t) = u(x, \pi, t) = 0, \quad 0 < x < \pi, \quad t > 0,
\]

\[
u(x, y, 0) = f(x, y), \quad 0 < x < \pi, \quad 0 < y < \pi,
\]

where

\[
f(x, y) = x \sin y.
\]

**Note:** Questions 15 and 18 concern the heat equation in higher dimensions (the method of Example 4 on pages 607–609). This topic was not covered in class, so these two questions should be regarded as optional.

We follow Example 4 on pages 607–609. For this problem, \( \beta = 1, \ L = W = \pi \), and \( f(x, y) = x \sin y \). We need a double Fourier series for \( f(x, y) = x \sin y \), in terms of \( \cos nx \) and \( \sin my \). The Fourier cosine series for the (single-variable) function \( x \) is

\[
a_n = \frac{2}{\pi} \int_{0}^\pi x \cos nx \, dx = \frac{2}{\pi} \int_{0}^\pi x \sin nx \, dx
\]

\[
= \frac{2x}{n\pi} \sin nx \bigg|_{0}^{\pi} - \frac{2}{n\pi} \int_{0}^{\pi} \sin nx \, dx
\]

\[
= 0 + \frac{2}{n\pi} \cos nx \bigg|_{0}^{\pi} - \frac{2}{n\pi} \cos nx \bigg|_{0}^{\pi}
\]

\[
= \begin{cases} 
\frac{-4}{n\pi}, & n \text{ odd} \\
0, & n \text{ even}
\end{cases}
\]
(using integration by parts) if \( n > 0 \). Therefore

\[
x \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x
\]

and therefore

\[
x \sin y \sim \frac{\pi}{2} \sin y - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x \sin y .
\]

This is the equivalent of (53) for this problem, so substituting the coefficients into (52) gives the formal solution

\[
u(x, y, t) = \frac{\pi}{2} e^{-t} \sin y - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} e^{-(2n-1)^2 t} \cos(2n-1)x \sin y .
\]

19. **Chemical diffusion.** Chemical diffusion through a thin layer is governed by the equation

\[
\frac{\partial C}{\partial t} = k \frac{\partial^2 C}{\partial x^2} - LC ,
\]

where \( C(x, t) \) is the concentration in moles/cm\(^3\), the diffusivity \( k \) is a positive constant with units cm\(^2/\)sec, and \( L > 0 \) is a consumption rate with units sec\(^{-1}\). Assume the boundary conditions are

\[
C(0, t) = C(a, t) = 0 , \quad t > 0 ,
\]

and the initial concentration is given by

\[
C(x, 0) = f(x) , \quad 0 < x < a .
\]

Use the method of separation of variables to solve formally for the concentration \( C(x, t) \). What happens to the concentration as \( t \to +\infty \)?

Substituting \( C = X(x)T(t) \) into the differential equation \( \frac{\partial C}{\partial t} = k \frac{\partial^2 C}{\partial x^2} - LC \) gives

\[
X(x)T''(t) = kX''(x)T(t) - LX(t)T(t) .
\]

Dividing by \( X(t)T(t) \) then gives

\[
\frac{T''(t)}{T(t)} = k \frac{X''(x)}{X(x)} - L ,
\]

and (as usual) since the left-hand side is independent of \( x \) and the right-hand side is independent of \( t \), the value depends on neither variable, so it must be constant. This gives equations

\[
kX'' - LX = -k\lambda X \quad \text{and} \quad T' = -k\lambda T .
\]
The first of these also has conditions \( X(0) = X(a) = 0 \). The equation may be rewritten as \( X'' + (\lambda - L/k)X = 0 \), so by the work on pages 571–572 we have eigenfunctions only when
\[
a \sqrt{\lambda - \frac{L}{k}} = n\pi, \quad n = 1, 2, 3, \ldots.
\]
Therefore
\[
\lambda = \left(\frac{n\pi}{a}\right)^2 + \frac{L}{k}, \quad n = 1, 2, 3, \ldots.
\]
Corresponding eigenfunctions are
\[
X(x) = \sin \frac{n\pi x}{a}.
\]

The differential equation \( T' = -k\lambda T \) has solution
\[
T(t) = ce^{-k\left(\frac{n\pi}{a}\right)^2 t}
\]
for a constant \( c \).

The solution to the problem, then, is that if
\[
b_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} \, dx, \quad n = 1, 2, 3, \ldots,
\]
then
\[
f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a},
\]
and a formal solution to the differential equation is
\[
C(x, t) = \sum_{n=1}^{\infty} b_n e^{-k\left(\frac{n\pi}{a}\right)^2 t} \sin \frac{n\pi x}{a}.
\]

As \( t \to \infty \), the concentration tends to 0 (due to the negative exponential factor in each term).

**Section 10.6 (Page 622)**

7. Find a formal solution to the vibrating string problem governed by the given nonhomogeneous initial-boundary value problem.

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + tx, \quad 0 < x < \pi, \quad t > 0,
\]
\[
u(0, t) = u(\pi, t) = 0, \quad t > 0,
\]
\[
u(x, 0) = \sin x, \quad 0 < x < \pi,
\]
\[
\frac{\partial u}{\partial t}(x, 0) = 5 \sin 2x - 3 \sin 5x, \quad 0 < x < \pi.
\]
This solution follows Example 1 on pages 614–615. We look for a solution of the form

\[ u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin nx. \]  \hspace{1cm} (1)

Since \( h(x,t) = tx \), we use the sine series from Exercise 9 on page 592 to get

\[ tx \sim \sum_{n=1}^{\infty} \frac{2t(-1)^{n+1}}{n} n \sin nx. \]

Formally substituting this into the main partial differential equation then gives

\[ \sum_{n=1}^{\infty} u_n''(t) \sin nx = -\sum_{n=1}^{\infty} n^2 u_n(t) \sin nx + \sum_{n=1}^{\infty} \frac{2t(-1)^{n+1}}{n} \sin nx. \]

Equating the coefficients in each series then gives

\[ u_n''(t) + n^2 u_n(t) = \frac{2t(-1)^{n+1}}{n} \]

for each \( n \geq 1 \). This is a non-homogeneous second-order linear differential equation with constant coefficients. The associated homogeneous differential equation has auxiliary polynomial is \( r^2 + n^2 \), which has roots \( \pm ni \). Therefore, the general solution of the associated homogeneous equation is \( c_{n,1} \cos nt + c_{n,2} \sin nt \).

To find a particular solution of the non-homogeneous equation in \( u_n(t) \), we use a trial solution \( u_n(t) = A_n t + B_n \). Substituting this into the differential equation gives

\[ n^2 (A_n t + B_n) = \frac{2t(-1)^{n+1}}{n} ; \]

equating coefficients of \( t \) and 1 gives \( B_n = 0 \) and \( A_n = 2(-1)^{n+1}/n^3 \), so the general solution of the equation in \( u_n(t) \) is

\[ u_n(t) = \frac{2(-1)^{n+1}}{n^3} t + c_{n,1} \cos nt + c_{n,2} \sin nt. \]  \hspace{1cm} (2)

Plugging (1) into the initial conditions \( u(x,0) = \sin x \) and

\[ \frac{\partial u}{\partial t}(x,0) = 5 \sin 2x - 3 \sin 5x \] gives

\[ \sum_{n=1}^{\infty} u_n(0) \sin nx = \sin x \quad \text{and} \quad \sum_{n=1}^{\infty} u'_n(0) \sin nx = 5 \sin 2x - 3 \sin 5x. \]
Equating terms then gives
\[ u_n(0) = \begin{cases} 1, & n = 1; \\ 0, & n \neq 1 \end{cases} \quad \text{and} \quad u'_n(0) = \begin{cases} 5, & n = 2; \\ -3, & n = 5; \\ 0, & n \neq 3, 5. \end{cases} \]

On the other hand, by (2), \( u_n(0) = c_{n,1} \) and \( u'_n(0) = \frac{2(-1)^{n+1}}{n^3} + nc_{n,2} \), so
\[ c_{n,1} = u_n(0) \quad \text{and} \quad c_{n,2} = \frac{2(-1)^n}{n^4} + \frac{u'_n(0)}{n}. \]

Combining these gives the formal solution
\[ u(x, t) = \cos t \sin x + \frac{5}{2} \sin 2t \sin 2x - \frac{3}{5} \sin 5t \sin 5x + 2 \sum_{n=1}^{\infty} (-1)^n \left( \frac{\sin nt}{n^4} - \frac{t}{n^3} \right) \sin nx. \]

Section 10.7 (Page 634)

5. Find a formal solution to the given boundary value problem.

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi, \quad 0 < y < 1, \]
\[ \frac{\partial u}{\partial x}(0, y) = \frac{\partial u}{\partial x}(\pi, y) = 0, \quad 0 \leq y \leq 1, \]
\[ u(x, 0) = \cos x - \cos 3x, \quad 0 \leq x \leq \pi, \]
\[ u(x, 1) = \cos 2x, \quad 0 \leq x \leq \pi. \]

We follow the method of Example 1 on page 625. We are using that example with \( a = \pi \) and \( b = 1 \), so (as in the example), separation of variables gives
\[ X_n(x) = a_n \cos nx \quad \text{and} \quad Y_n(y) = A_n \cosh ny + B_n \sinh ny. \]

We need to find \( A_n \) and \( B_n \) for \( n = 1, \ n = 2, \) and \( n = 3 \).

For \( n = 1 \), we need \( Y_1(0) = 1 \) (since \( \cos x \) appears in the boundary value for \( u(x, 0) \) with coefficient 1), and \( Y_1(1) = 0 \). As in the middle of page 626, this leads to \( Y_1(y) = c_1 \sinh(y - 1) \). The boundary value \( Y_1(0) = 1 \) then gives \( c_1 = 1/\sinh(-1) \), so
\[ Y_1(y) = -\frac{\sinh(y - 1)}{\sinh 1}. \]

For \( n = 2 \), we need \( Y_2(0) = 0 \) and \( Y_2(1) = 1 \), so in the above expression for \( Y_N \) the first condition leads to \( A_2 = 0 \), and the second gives \( B_2 = 1/\sinh 2 \); hence
\[ Y_2(y) = \frac{\sinh 2y}{\sinh 2}. \]
Finally, for $n = 3$ we need $Y_3(0) = -1$ and $Y_3(1) = 0$, and as in the $n = 1$ case this leads to $Y_3(y) = B_3 \sinh 3(y - 1)$, with $B_3 = -1/\sinh(-3) = 1/\sinh 3$. Thus

$$Y_3(y) = \frac{\sinh 3(y - 1)}{\sinh 3}.$$ 

Therefore, the solution is

$$u(x, y) = -\frac{\cos x \sinh(y - 1)}{\sinh 1} + \frac{\cos 2x \sinh 2y}{\sinh 2} + \frac{\cos 3x \sinh 3(y - 1)}{\sinh 3}.$$