MATH 54 – QUIZ 9 – SOLUTIONS

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1. *(5 points)* Find a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ (where a and b are **real** numbers) and an invertible matrix P such that $A = PCP^{-1}$, where

$$A = \begin{bmatrix} 3 & -5 \\ 2 & 5 \end{bmatrix}$$

<u>Eigenvalues:</u>

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & 5 \\ -2 & \lambda - 5 \end{vmatrix} = (\lambda - 3)(\lambda - 5) + 10 = \lambda^2 - 8\lambda + 25 = 0$$

Using the quadratic formula, we get:

$$\lambda = \frac{8 \pm \sqrt{64 - 100}}{2} = \frac{8 \pm \sqrt{-36}}{2} = \frac{8 \pm 6i}{2} = 4 \pm 3i$$

From now on, consider the eigenvalue $\lambda = 4 + 3i$, then:

$$C = \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$$

(remember that the first **row** of C consists of the real and imaginary parts of λ)

<u>Eigenvectors:</u>

<u>$\lambda = 4 + 3i$</u>

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$$
\begin{aligned}
Nul((4+3i)I - A) =& Nul \begin{bmatrix} (4+3i)-3 & 5 \\ -2 & (4+3i)-5 \end{bmatrix} \\
=& Nul \begin{bmatrix} 1+3i & 5 \\ -2 & -1+3i \end{bmatrix} \\
=& Nul \begin{bmatrix} 1 & \frac{5}{1+3i} \\ 1 & \frac{-1+3i}{-2} \end{bmatrix} \\
=& Nul \begin{bmatrix} 1 & \frac{1-3i}{2} \\ 1 & \frac{1-3i}{2} \end{bmatrix} \\
=& Nul \begin{bmatrix} 1 & \frac{1-3i}{2} \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

In the fourth line, we used:

$$\frac{5}{1+3i} = \frac{5(1-3i)}{(1+3i)(1-3i)} = \frac{5(1-3i)}{1+9} = \frac{5(1-3i)}{10} = \frac{1-3i}{2}$$

Now if:

$$\begin{bmatrix} 1 & \frac{1-3i}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then $x + \left(\frac{1-3i}{2}\right) y = 0$, so $x = -\left(\frac{1-3i}{2}\right) y$, and so:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\left(\frac{1-3i}{2}\right) y \\ y \end{bmatrix} = y \begin{bmatrix} -\frac{1}{2} + \frac{3}{2}i \\ 1 \end{bmatrix}$$

And therefore a basis for the eigenspace corresponding to $\lambda = 4+3i$ is $\left\{ \begin{bmatrix} -\frac{1}{2} + \frac{3}{2}i \\ 1 \end{bmatrix} \right\} \sim \left\{ \begin{bmatrix} -1+3i \\ 2 \end{bmatrix} \right\}$ (remember it's ok to multiply your eigenvector by a nonzero constant).

Let $\mathbf{u} = \begin{bmatrix} -1+3i \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + i \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Then:

$$P = \begin{bmatrix} Re(\mathbf{u}) & Im(\mathbf{u}) \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$$

Hence:

$$C = \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix} \quad P = \begin{bmatrix} Re(\mathbf{u}) & Im(\mathbf{u}) \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$$

Note: If you considered $\lambda = 4 - 3i$, you would have gotten an equally acceptable answer:

$$C = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \quad P = \begin{bmatrix} -1 & -3 \\ 2 & 0 \end{bmatrix}$$

2. *(5 points)* Find all vectors $\mathbf{x}$ in $\mathbb{R}^4$ which are (simultaneously) orthogonal to:

$$\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Suppose $\mathbf{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. Then we know:

$$\mathbf{x} \cdot \mathbf{u} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \cdot \mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = b + c = 0$$

$$\mathbf{x} \cdot \mathbf{v} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \cdot \mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = a + c = 0$$

$$\mathbf{x} \cdot \mathbf{w} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \cdot \mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = a + d = 0$$

In other words, we are led to solve the system:

$$\begin{cases} b+c = & 0 \\ a+c = & 0 \\ a+d = & 0 \end{cases}$$

Forming the augmented matrix (as usual) and row-reducing until we get the RREF, we get:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

which implies:

$$\begin{cases} a+d = & 0 \\ b+d = & 0 \\ c-d = & 0 \end{cases}$$

So $a = -d$, $b = -d$, $c = d$, and hence:

$$\mathbf{x} = \begin{bmatrix} -d \\ -d \\ d \\ d \end{bmatrix} = d \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = Span \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$