MATH 54 – QUIZ 3 – SOLUTIONS

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1. *(5 points)* Are the columns of the following matrix linearly independent or linearly dependent?

$$\begin{bmatrix} -4 & -3 & 15 \\ 0 & -1 & 5 \\ 1 & 1 & -5 \\ 2 & 1 & -10 \end{bmatrix}$$

We want to solve the system $A\mathbf{x} = \mathbf{0}$, where A is the matrix above. The augmented matrix becomes:

$$\begin{bmatrix} -4 & -3 & 15 & 0 \\ 0 & -1 & 5 & 0 \\ 1 & 1 & -5 & 0 \\ 2 & 1 & -10 & 0 \end{bmatrix}$$

Row-reducing until the matrix is in REF, we get:

$$\begin{bmatrix} 1 & 1 & -5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Fast way: There are 3 pivots in the coefficient matrix, hence the number of pivots (in the coefficient matrix) is equal to the number columns (in the coefficient matrix), and hence the columns of the matrix are $\boxed{\text{linearly independent}}$.

Slow way: Row-reducing further until we get the RREF:

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$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

But this implies that $x = 0, y = 0, z = 0$, and hence $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Therefore, the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$, and therefore the columns of the matrix are $\boxed{\text{linearly independent}}$

2. *(5 points)* Let $T(x, y) = (x + y, x - 2y)$.

(a) *(2 points)* Show that T is a linear transformation.

Slow way: Let $\mathbf{u} = (x, y)$ and $\mathbf{v} = (x', y')$, and c a constant.

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) =& T(x + x', y + y') \\ =& ((x + x') + (y + y'), (x + x') - 2(y + y')) \\ =& ((x + y) + (x' + y'), (x - 2y) + (x' - 2y')) \\ =& (x + y, x - 2y) + (x' + y', x' - 2y') \\ =& T(x, y) + T(x', y') \\ =& T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

$$\begin{aligned} T(c\mathbf{u}) =& T(cx, cy) \\ =& (cx + cy, cx - 2cy) \\ =& c(x + y, x - 2y) \\ =& cT(x, y) \\ =& cT(\mathbf{u}) \end{aligned}$$

Hence $\boxed{T \text{ is a linear transformation}}$

Fast way:

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}, T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix}$$

So if we let:

$$A = \begin{bmatrix}1 & 0\\1 & -2\end{bmatrix}$$

Then $T(\mathbf{x}) = A\mathbf{x}$. However, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is always a linear transformation, therefore $\boxed{T \text{ is a linear transformation}}$

(b) *(2 points)* Is T one-to-one?

Slow way: Suppose $T(x, y) = (0, 0)$, then: $(x + y, x - 2y) = (0, 0)$, hence $x + y = 0$ and $x - 2y = 0$.

From the second equation, we get $x = 2y$, and plugging this in to the first equation, we have $2y + y = 0$, so $3y = 0$, so $y = 0$, and hence $x = 2y = 0$.

Therefore $(x, y) = (0, 0)$, and hence $\boxed{T \text{ is one-to-one}}$

Fast way: Let's row-reduce the matrix A above (what else?):

$$\begin{bmatrix}1 & 0\\1 & -2\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0\\0 & -2\end{bmatrix}$$

Notice that A has 2 pivots, which is equal to the number of **columns** of A, and therefore the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$, and therefore $T(\mathbf{x}) = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0}$, and therefore $\boxed{T \text{ is one-to-one}}$

(c) *(1 point)* Does T map $\mathbb{R}^2$ **onto** $\mathbb{R}^2$?

Slow way: (not recommended) Let (a, b) be an arbitrary vector in $\mathbb{R}^2$. We want to find (x, y) such that $T(x, y) = (a, b)$.

But this means that $(x+y, x-2y) = (a, b)$, and hence $x+y = a$ and $x - 2y = b$. The second equation implies that $x = b + 2y$, and plugging this into the first, we get $y = a - x = a - (b + 2y) = a - b - 2y$. Solving for y, we get $3y = a - b$, so $y = \frac{a-b}{3}$, and finally $x = b + 2y = b + \frac{2}{3}(a - b) = \frac{2a+b}{3}$.

Hence if we let $(x, y) = \left(\frac{2a+b}{3}, \frac{a-b}{3}\right)$, then in fact $T(x, y) = (a, b)$. Therefore $\boxed{T \text{ is onto } \mathbb{R}^2}$

Fast way: Notice that in the matrix in (b), there are as many pivots are **rows**, and therefore for every $\mathbf{b}$ in $\mathbb{R}^2$, the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution, and therefore for every $\mathbf{y}$ in $\mathbb{R}^2$, there is at least one $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{y}$, and therefore $\boxed{T \text{ is onto } \mathbb{R}^2}$