MATH 54 – QUIZ 13 – SOLUTIONS

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1. *(5 points)* Find the general solution to the following differential equation:

$$\left(\frac{d^2}{dt^2}+1\right)\left(\frac{d}{dt}+2\right)^2\left(\frac{d^2}{dt^2}+4\frac{d}{dt}+5\right)y=0$$

The auxiliary equation is:

$$(r^2+1)(r+2)^2(r^2+4r+5)=0$$

which gives us $r^2+1=0 \Rightarrow r=\pm i$, $r=-2$ (double root), and $r^2+4r+5=0 \Rightarrow r=-2\pm i$, therefore the general solution to the differential equation is:

$$y(t)=A\cos(t)+B\sin(t)+Ce^{-2t}+Dte^{-2t}+Ee^{-2t}\cos(t)+Fe^{-2t}\sin(t)$$

Note: It would be incorrect to write $Ete^{-2t}\cos(t)$, or $At\cos(t)$. You should really view the root $-2\pm i$ as a whole. In particular, it does not coincide with either $r=-2$ or $r=\pm i$.

(TURN PAGE)

Date: Friday, December 5, 2014.

2. *(5 points)* Let:

$$\mathbf{x_1} = \begin{bmatrix} e^t \\ e^t \end{bmatrix}, \mathbf{x_2} = \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix}$$

Determine if $\{\mathbf{x_1}, \mathbf{x_2}\}$ form a fundamental solution set of the system:

$$\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x}$$

What is the general solution of the above system?

Check: First we need to check whether $\mathbf{x_1}$ and $\mathbf{x_2}$ satisfy our system:

$$\mathbf{x_1}' = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x_1} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} e^t \\ e^t \end{bmatrix} = \begin{bmatrix} 2e^t - e^t \\ 3e^t - 2e^t \end{bmatrix} = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$$

Therefore $\mathbf{x_1}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x_1}$.

Similarly:

$$\mathbf{x_2}' = \begin{bmatrix} -e^{-t} \\ -3e^{-t} \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x_2} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix} = \begin{bmatrix} 2e^{-t} - 3e^{-t} \\ 3e^{-t} - 6e^{-t} \end{bmatrix} = \begin{bmatrix} -e^{-t} \\ -3e^{-t} \end{bmatrix}$$

So $\mathbf{x_2}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x_2}$.

Therefore $\mathbf{x_1}$ and $\mathbf{x_2}$ solve our system of differential equations.

Linear independence: Next, we need to check if $\mathbf{x_1}$ and $\mathbf{x_2}$ are linearly independent. In this case, the Wronskian matrix is:

$$\widetilde{W}(t) = \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}$$

Choosing for example $t = 0$, we get:

$$\widetilde{W}(0) = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$

Then $W(0) = 3 - 1 = 2 \neq 0$, therefore $\mathbf{x_1}$ and $\mathbf{x_2}$ are linearly independent.

General solution: Therefore $\mathbf{x_1}$ and $\mathbf{x_2}$ are two linearly independent solutions to our system, and hence they form a solution set for our system.

In particular, it follows that the general solution to our system is:

$$\mathbf{x}(t) = A\mathbf{x_1}(t) + B\mathbf{x_2}(t) = A \begin{bmatrix} e^t \\ e^t \end{bmatrix} + B \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix}$$