MATH 54 – QUIZ 12 – SOLUTIONS

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1. *(5 points)* Find a particular solution to the following differential equation:

$$y'' - 3y' + 2y = 4te^{3t}$$

First of all, the auxiliary equation is $r^2 - 3r + 2 = 0$, that is $(r-1)(r-2) = 0$, so $r = 1, 2$, which means that the general solution of the homogeneous equation is:

$$y_0(t) = Ae^t + Be^{2t}$$

Now to guess the particular solution, treat the t-term and the $4e^{3t}$-term separately:

t becomes $(At + B)$ (since this does not coincide with y_0), and $4e^{3t}$ becomes Ae^{3t} (since this does not coincide with y_0), and therefore you guess:

$$y_p(t) = (At + B)e^{3t}$$

This gives us:

$$\begin{aligned} y_p' =& Ae^{3t} + 3(At + B)e^{3t} = (3At + A + 3B)e^{3t} \\ y_p'' =& 3Ae^{3t} + (9At + 3A + 9B)e^{3t} = (9At + 6A + 9B)e^{3t} \end{aligned}$$

Now plug this into our equation:

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$$\begin{aligned}(y_p)'' - 3(y_p)' + 2(y_p) &= 4te^{3t} \\ (9At + 6A + 9B)e^{3t} - 3(3At + A + 3B)e^{3t} + 2(At + B)e^{3t} &= 4te^{3t} \\ (9At + 6A + 9B)e^{3t} - (9At + 3A + 9B)e^{3t} + (2At + 2B)e^{3t} &= 4te^{3t} \\ (9A - 9A + 2A)te^{3t} + (6A + 9B - 3A - 9B + 2B)e^{3t} &= 4te^{3t} \\ 2Ate^{3t} + (3A + 2B)e^{3t} &= 4te^{3t} + 0e^{3t}\end{aligned}$$

Comparing the left-hand-side with the right-hand-side, we get:

$$\begin{cases} 2A = 4 \\ 3A + 2B = 0 \end{cases} \Rightarrow \begin{cases} A = 2 \\ B = -\frac{3}{2}A = -3 \end{cases}$$

And therefore a particular solution to our differential equation is:

$$y_p(t) = (At + B)e^{3t} = (2t - 3)\, e^{3t}$$

2. *(5 points)* Find the general solution of the following differential equation:

$$y'' - 4y' + 5y = e^{2t} + 8\sin(t)$$

Homogeneous solution:

The auxiliary equation is $r^2 - 4r + 5 = 0$, which gives us $r = \frac{4\pm\sqrt{-4}}{2} = \frac{4\pm 2i}{2} = 2 \pm i$, and therefore the general solution to $y'' - 4y' + 5y = 0$ is:

$$y_0(t) = Ae^{2t}\cos(t) + Be^{2t}\sin(t)$$

Particular solution:

Using the superposition principle, first find y_1, where y_1 is a particular solution to:

$$(y_1)'' - 4(y_1)' + 5(y_1) = e^{2t}$$

For this, guess $y_1(t) = Ae^{2t}$ (this is ok because y_1 does **not** coincide with y_0). Then we get:

$$\begin{aligned}(Ae^{2t})'' - 4(Ae^{2t})' + 5(Ae^{2t}) &= e^{2t}\\ 4Ae^{2t} - 8Ae^{2t} + 5Ae^{2t} &= e^{2t}\\ Ae^{2t} &= e^{2t}\end{aligned}$$

which gives $A = 1$, and so $\boxed{y_1(t) = e^{2t}}$

And now find y_2, where y_2 is a particular solution to:

$$(y_2)'' - 4(y_2)' + 5(y_2) = 8\sin(t)$$

For this, guess $y_2(t) = A\cos(t) + B\sin(t)$ (this is also ok because y_2 doe not coincide with y_0). Then we get:

$$\begin{aligned}(A\cos(t) + B\sin(t))'' - 4(A\cos(t) + B\sin(t))' + 5(A\cos(t) + B\sin(t)) &= 8\sin(t)\\ -A\cos(t) - B\sin(t) + 4A\sin(t) - 4B\cos(t) + 5A\cos(t) + 5B\sin(t) &= 8\sin(t)\\ (-A - 4B + 5A)\cos(t) + (-B + 4A + 5B)\sin(t) &= 8\sin(t)\\ (4A - 4B)\cos(t) + (4B + 4A)\sin(t) &= 0\cos(t) + 8\sin(t)\end{aligned}$$

which gives us:

$$\begin{cases} 4A - 4B = 0 \\ 4B + 4A = 8 \end{cases} \Rightarrow \begin{cases} A = B \\ 8B = 8 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = 1 \end{cases}$$

And therefore $\boxed{y_2(t) = \cos(t) + \sin(t)}$.

And therefore a particular solution to our differential equation is:

$$y_p(t) = y_1(t) + y_2(t) = e^{2t} + \cos(t) + \sin(t)$$

And therefore the *general solution* to our differential equation is:

$$y(t) = y_0(t) + y_p(t) = Ae^{2t}\cos(t) + Be^{2t}\sin(t) + e^{2t} + \cos(t) + \sin(t)$$