MATH 54 – QUIZ 11 – SOLUTIONS

PEYAM RYAN TABRIZIAN

1. *(5 points)* Find a diagonal matrix D and an **orthogonal** matrix P such that $A = PDP^T$, where:

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

We are given that the eigenvalues of A are $\lambda = -2$ and $\lambda = 7$

$\underline{\lambda = -2}$

We are given that a basis for E_{-2} is $\left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right\}$

Let $\mathbf{u_1} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. Applying Gram-Schmidt to $\mathbf{u_1}$, we get:

$\mathbf{v_1} = \mathbf{u_1}$, and $\mathbf{w_1} = \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

Therefore an orthonormal basis for E_{-2} is $\left\{ \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix} \right\}$

$\underline{\lambda = 7}$

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$$\begin{aligned}
Nul(7I - A) =& Nul \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix} \\
=& Nul \begin{bmatrix} 2 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
=& Span \left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \\
=& Span \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}
\end{aligned}$$

Let $\mathbf{u_1} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and apply Gram-Schmidt to $\mathbf{u_1}$ and $\mathbf{u_2}$:

$$\mathbf{v_1} = \mathbf{u_1} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$\hat{\mathbf{u_2}} = \left(\frac{\mathbf{u_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}} \right) \mathbf{v_1} = \frac{1}{5} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ -\frac{2}{5} \\ 0 \end{bmatrix}$$

$$\mathbf{v_2} = \mathbf{u_2} - \hat{\mathbf{u_2}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{5} \\ -\frac{2}{5} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{2}{5} \\ 1 \end{bmatrix} \sim \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

$$\mathbf{w_1} = \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|} = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \\ 0 \end{bmatrix}$$

$$\mathbf{w_2} = \frac{\mathbf{v_2}}{\|\mathbf{v_2}\|} = \frac{1}{\sqrt{45}} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4/\sqrt{45} \\ 2/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix}$$

Therefore an orthonormal basis for E_7 is:

$$\left\{ \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 4/\sqrt{45} \\ 2/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix} \right\}$$

Answer:

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \quad P = \begin{bmatrix} 2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ 1/3 & -2/\sqrt{5} & 2/\sqrt{45} \\ -2/3 & 0 & 5/\sqrt{45} \end{bmatrix}$$

Notice that you can check your answer by checking that the columns of P are indeed orthonormal or not!

2. *(5 points)* Solve the following differential equation:

$$\begin{cases} y'' - 2y + 5y = 0 \\ y(0) = 2 \\ y'(0) = 8 \end{cases}$$

Auxiliary equation:

$$r^2 - 2r + 5 = 0 \Rightarrow r = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$$

General solution:

Therefore the general solution of $y'' - 2y' + 5y = 0$ is:

$$y(t) = Ae^t \cos(2t) + Be^t \sin(2t)$$

Solving for A and B:

Plugging in $t = 0$, we get:

$$y(0) = A + 0 = A = 2$$

and hence $A = 2$, and $y(t) = 2e^t \cos(2t) + Be^t \sin(2t)$.

Now differentiating, we get:

$$
\begin{aligned}
y'(t) = 2e^t \cos(2t) - 4e^t \sin(2t) + Be^t \sin(2t) + 2Be^t \cos(2t) \\
= (2 + 2B)e^t \cos(2t) + (-4 + B)e^t \sin(2t)
\end{aligned}
$$

Plugging in $t = 0$, we get:

$$y'(0) = (2 + 2B) + 0 = 6 \Rightarrow 2B = 8 - 2 = 6 \Rightarrow B = \frac{6}{2} = 3$$

And hence $A = 2$, $B = 3$, and so the answer is:

$$y(t) = 2e^t \cos(2t) + 3e^t \sin(2t)$$