MATH 54 – QUIZ 10 – SOLUTIONS

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1. *(5 points)* Find an orthonormal basis for $W = Span\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$, where:

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{v_1} = \mathbf{u_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{v_2} = \mathbf{u_2} - \left(\frac{\mathbf{u_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right)\mathbf{v_1} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

$$\mathbf{v_3} = \mathbf{u_3} - \left(\frac{\mathbf{u_3} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right)\mathbf{v_1} - \left(\frac{\mathbf{u_3} \cdot \mathbf{v_2}}{\mathbf{v_2} \cdot \mathbf{v_2}}\right)\mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 1 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

Hence:

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

Finally, normalize to get that an orthonormal basis for W is $\mathcal{B} = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$, where:

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$$\mathbf{w_1} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \mathbf{w_2} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-2\\-1\\0\end{bmatrix}, \mathbf{w_3} = \frac{1}{\sqrt{12}}\begin{bmatrix}1\\1\\-1\\3\end{bmatrix}$$

2. *(5 points)* Find the least-squares solution of $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{bmatrix}2 & 1\\1 & 0\\1 & -2\end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

Hint: If it helps you, notice that the columns of A are orthogonal.

OMG-way: Since the columns of A are orthogonal, let's calculate:

$$\begin{aligned}\hat{\mathbf{b}} &= \left(\frac{\begin{bmatrix}1\\0\\1\end{bmatrix}\cdot\begin{bmatrix}2\\1\\1\end{bmatrix}}{\begin{bmatrix}2\\1\\1\end{bmatrix}\cdot\begin{bmatrix}2\\1\\1\end{bmatrix}}\right)\begin{bmatrix}2\\1\\1\end{bmatrix} + \left(\frac{\begin{bmatrix}1\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\0\\-2\end{bmatrix}}{\begin{bmatrix}1\\0\\-2\end{bmatrix}\cdot\begin{bmatrix}1\\0\\-2\end{bmatrix}}\right)\begin{bmatrix}1\\0\\-2\end{bmatrix}\\ &= \frac{3}{6}\begin{bmatrix}2\\1\\1\end{bmatrix} + \frac{-1}{5}\begin{bmatrix}1\\0\\-2\end{bmatrix}\\ &= \left(\frac{1}{2}\right)\begin{bmatrix}2\\1\\1\end{bmatrix} + \left(-\frac{1}{5}\right)\begin{bmatrix}1\\0\\-2\end{bmatrix}\end{aligned}$$

Then the least-squares solution of $A\mathbf{x} = \mathbf{b}$, which is the solution of $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$, is given by the coefficients of the linear combination of $\hat{\mathbf{b}}$ above, and therefore:

$$\hat{\mathbf{x}} = \begin{bmatrix}\frac{1}{2}\\-\frac{1}{5}\end{bmatrix}$$

(Alternatively, just solve $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$ using row-reduction)

<u>Standard way:</u>

Solve the system:

$$A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -2 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 0 \\ 0 & 5 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

which gives:

$$\hat{\mathbf{x}} = \begin{bmatrix} \frac{3}{6} \\ -\frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{5} \end{bmatrix}$$