SYSTEMS OF EQUATIONS – TWO POINTS OF VIEW

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Here are notes to special make-up discussion 3 on September 4, in case you couldn't make it.

Note: To make those notes more legible, I'm skipping the row-reduction parts.

Admin stuff:

- Quiz 2 tomorrow at $12 : 10$ pm; remember you can check your answer by plugging in
- Homeworks are graded out of 5. Make sure to attempt every problem, and **show your work**

Yesterday, we learned a very powerful method called row-reduction, which allows us to solve *any* system of equations. Today we'll go back to the original system and ask ourselves: "what *is* a system of equations, really?" We'll learn two ways of rewriting a system which will be useful later on. You might say "why bother?" The point is that once we rewrite the syste we'll be able to generalize this to things which are not systems of equations (like... differential equations!)

1. THE MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

Example: [1.4.12] Rewrite the following system in the form $A\mathbf{x} = \mathbf{b}$, and solve it:

$$\begin{cases} x + 2y - z = 1 \\ -3x - 4y + 2z = 2 \\ 5x + 2y + 3z = -3 \end{cases}$$

This is nothing other than row-reduction; namely, let:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -4 & 2 \\ 5 & 2 & 3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

Date: Thursday, September 4, 2014.

First point of view of systems: Then the system is of the form $A\mathbf{x} = \mathbf{b}$, that is:

$$\begin{bmatrix} 1 & 2 & -1 \\ -3 & -4 & 2 \\ 5 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

(Why? You'll have to wait until Chapter 2 to find this out, because in chapter 2 we'll define matrix multiplication, and you'll see that this is just a consequence of matrix multiplication)

The advantage of writing is this way is that it's easy to write the augmented matrix:

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ -3 & -4 & 2 & 2 \\ 5 & 2 & 3 & -3 \end{bmatrix}$$

And row-reducing (I'm skipping this step here), we get:

$$\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This tells you that $x = -4, y = 4, z = 3$, and so:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 3 \end{bmatrix}$$

We'll see later on that this is really *the* correct point of view that we'll adopt in this class. So from now on whenever we'll talk about systems, we'll always say $A\mathbf{x} = \mathbf{b}$

Food for thought: Since $A\mathbf{x} = \mathbf{b}$, could you just say $\mathbf{x} = A^{-1}\mathbf{b}$ (that would be nice since you'll find $\mathbf{x}$ right away). And in fact, in most cases this does make sense, but we'll have to wait until chapter 2.

Example: Solve $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$$

Again, form the augmented matrix and row-reduce until you get the **reduced** row-echelon form:

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This tells us that:

$$\begin{cases} x + 5z = 2 \\ y + 4z = 3 \end{cases} \implies \begin{cases} x = 2 - 5z \\ y = 3 - 4z \\ z = z \end{cases}$$

Which tells us that:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 - 5z \\ 3 - 4z \\ 0 + z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + z \begin{bmatrix} -5 \\ -4 \\ 1 \end{bmatrix}$$

(where $z \in \mathbb{R}$; this is called the parametric vector form)

Geometric interpretation: This just says that the solution is a line in $\mathbb{R}^3$ going through the point $(2, 3, 0)$ and with "slope" $(-5, -4, 1)$.

2. Linear combinations and Span

The idea is that given vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$, what other vectors can you form?

For example, you can form $\mathbf{u} + \mathbf{v} + \mathbf{w}$, or $2\mathbf{u}$, or $5\mathbf{u} - 3\mathbf{v} + 7\mathbf{w}$.

More generally:

Definition: A **linear combination** of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is an expression of the form: $a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$, where $a, b, c \in \mathbb{R}$.

Example: Is $\begin{bmatrix} 11 \\ -5 \\ 22 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}$?

In other words, are there constants a, b, c such that:

$$a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} + c \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ 22 \end{bmatrix}$$

Writing this out, you'll see that this is the same as solving the system $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 7 \\ 1 & -2 & 5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 11 \\ -5 \\ 22 \end{bmatrix}$$

Row-reducing, we get:

$$\begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 22 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 11 & 11 \end{bmatrix}$$

Now there are two ways of concluding:

Slow way: Either continue row-reducing until you find $a = 9, b = -4, c = 1$, and therefore the answer is **YES**:

$$\begin{bmatrix} 11 \\ -5 \\ 22 \end{bmatrix} = 9 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (-4) \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} + (1) \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}$$

Fast way: Notice that we don't really care what a, b, c are. What we really case about is if a, b, c exist; in other words, all we really care about is if the system is **consistent** (or not!). For this, notice that in the REF above, there are 3 pivots in the coefficient matrix, hence no rows of the form $\begin{bmatrix} 0 & 0 & 0 & \star \end{bmatrix}$, where $\star \neq 0$, and therefore the system is consistent. And therefore the answer is **YES**.

This gives us another:

Useful test: If there are as many pivots in the **coefficient** matrix as there are **columns**, then the system is consistent.

(but this doesn't mean that if a matrix has only, say, 2 pivots in the coeff. matrix, then the system is inconsistent)

Moral: (of this example) $A\mathbf{x} = \mathbf{b}$ means: Is $\mathbf{b}$ a linear combo of the columns of A?

Which leads us to the:

Second point of view of systems:

$$\begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 7 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ 22 \end{bmatrix}$$

is really the same as:

$$x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} + z \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ 22 \end{bmatrix}$$

(so x, y, z are the weights of the linear combination; in other words, how do you get the vector on the right-hand-side using only the vectors on the left-hand-side and linear combinations)

Span: Linear combinations naturally lead to the idea of Span:

Definition: Span $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is the set of **all** linear combinations of $\mathbf{u}, \mathbf{v}, \mathbf{w}$:

$$\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \{a\mathbf{u} + b\mathbf{v} + c\mathbf{w} \mid a, b, c \in \mathbb{R}\}$$

(in general, Span is a HUGE set, usually infinitely big, whereas $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ has 3 elements)

Example: Is $\begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix}$ in:

$$\text{Span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} \right\}$$

In other words, are there numbers a, b, c such that:

$$a \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix} + c \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix}$$

In other words, is the following system consistent?

$$\begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix}$$

Row-reducing, you get:

$$\begin{bmatrix} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

The last row indicates that the system is inconsistent, therefore the answer is $\boxed{\text{NO}}$.