DIFFERENTIAL EQUATIONS – REVIEW

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Here are notes to special make-up discussion 35 on November 21, in case you couldn't make it.

Welcome to the special Friday after-school special of "That's so Peyam!" I realize that I've been going through Chapters 4, 6, and 9 like a rocket, so now we can finally slow down and ask ourselves: "What in the world are we doing?"

But first, let me wrap up with one last cute topic, which is **NOT** on the exam, but which is **VERY** important in case you want to study more differential equations in the future.

1. The matrix exponential

Idea: The idea is to go back to the roots (lol, roots). Since the general solution of $y' = ay$ (where a is in $\mathbb{R}$) is $y(t) = Ce^{at}$, why not just say that the general solution of $\mathbf{x}' = A\mathbf{x}$ is:

$$\mathbf{x}(t) = \mathbf{c}e^{At}$$

where $\mathbf{c}$ is a vector of constants.

Well, this doesn't quite make sense since $\mathbf{c}$ is an $n \times 1$ matrix, and e^{At} (should be) an $n \times n$ matrix, and the product of an $n \times 1$ and an $n \times n$ matrix is undefined. So we need to reformulate our question:

Question: Can we say that:

$$\mathbf{x}(t) = e^{At}\mathbf{c}$$

This would indeed provide us with an elegant way of solving $\mathbf{x}' = A\mathbf{x}$, and indeed we can! And as usual, the answer is... diagonalization!

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Example: Calculate e^{At}, where $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$, and use this to solve $\mathbf{x}' = A\mathbf{x}$.

The trick is to diagonalize A: Write $A = PDP^{-1}$, where:

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Notice that using diagonalization, it is easy to calculate A^{anything}. For example $A^2 = PD^2P^{-1}$, $A^3 = PD^3P^{-1}$, $A^n = PD^nP^{-1}$, and D^{anything} is even easier to calculate. For example, $D^n = \begin{bmatrix} (-1)^n & 0 \\ 0 & 1^n \end{bmatrix}$.

Using this idea, it makes sense (and indeed it is true) to say that $e^A = Pe^DP^{-1}$, and in general:

$$\begin{aligned} e^{At} =& Pe^{Dt}P^{-1} \\ =& \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \\ =& \begin{bmatrix} e^{-t} & e^t - e^{-t} \\ 0 & e^t \end{bmatrix} \end{aligned}$$

And therefore, if $\mathbf{c} = \begin{bmatrix} A \\ B \end{bmatrix}$ (where A and B are constants), we obtain:

$$\begin{aligned} \mathbf{x}(t) =& e^{At}\mathbf{c} \\ =& \begin{bmatrix} e^{-t} & e^t - e^{-t} \\ 0 & e^t \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \\ =& A \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix} + B \begin{bmatrix} e^t - e^{-t} \\ e^t \end{bmatrix} \end{aligned}$$

Note: Compare this with the solution $\mathbf{x}(t) = Ae^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + Be^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we would have obtained using diagonalization. At first sight it does not look like we obtained the same solution, but in fact, if we relabel the constants A and B, one can show that those two are the same solution.

Of course, so far it looks like that there is no difference between the (usual) diagonalization-way and the (new) matrix exponential-way, but using this idea of matrix exponentials, we can go *beyond* solving $\mathbf{x}' = A\mathbf{x}$. Get ready for the next example!

Example: With A as above, solve $\mathbf{x}'' = -A^2\mathbf{x}$.

Whoa! What a complicated system!!! **YET** we can solve this using the ideas of the previous example. Recall that the solutions of $y'' = -a^2 y$ are $y(t) = c_1 \cos(at) + c_2 \sin(at)$. Here, by analogy, we get:

$$\begin{aligned}
\mathbf{x}(t) &= \cos(At)\mathbf{c_1} + \sin(At)\mathbf{c_2} \\
&= \cos(At)\begin{bmatrix} A \\ B \end{bmatrix} + \sin(At)\begin{bmatrix} C \\ D \end{bmatrix} \\
&= P\cos(Dt)P^{-1}\begin{bmatrix} A \\ B \end{bmatrix} + P\sin(Dt)P^{-1}\begin{bmatrix} C \\ D \end{bmatrix} \\
&= P\begin{bmatrix} \cos(t) & 0 \\ 0 & \cos(-t) \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} + P\begin{bmatrix} \sin(t) & 0 \\ 0 & \sin(-t) \end{bmatrix}P^{-1}\begin{bmatrix} C \\ D \end{bmatrix} \\
&= \cdots
\end{aligned}$$

And in the end, you can find an explicit formula for $\mathbf{x}(t)$ in terms of $\cos$ and $\sin$ (I'll leave it up to you to finish the calculations)

Congratulations! Now, in theory, you can really solve *all* systems of differential equations!

2. Higher-Order Differential Equations

Example: Find all the solutions to $y'''' - 5y''' - 2y'' + 24y' = 0$ such that $\lim_{t\to\infty} y(t) = 54$.

Just as usual, find the auxiliary equation:

<u>Aux:</u> $r^4 - 5r^3 - 2r^2 + 24r = r(r^3 - 5r^2 - 2r + 24) = 0$.

In order to factor out $r^3 - 5r^2 - 2r + 24$, use the rational roots theorem, which says that if $r = \frac{a}{b}$, then a must divide 24 (the constant term) and b must divide 1 (the leading term). This gives us that $a = \pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24$, and $b = \pm1$, so we must try:

$$r = \pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24$$

(Good luck!)

After some calculations, you'll find that $\boxed{r=-2}$ works! (remember to check negative values as well, your roots are not always positive).

Now use long-division to divide $r^3 - 5r^2 - 2r + 24$ by $r+2$:

$$
\begin{array}{r|rrr}
 & X^2 & -7X & +12 \\ \hline
X+2 & X^3 - 5X^2 & -2X & +24 \\
 & -X^3 - 2X^2 & & \\ \hline
 & -7X^2 & -2X & \\
 & 7X^2 & +14X & \\ \hline
 & & 12X & +24 \\
 & & -12X & -24 \\ \hline
 & & & 0
\end{array}
$$

And you obtain that the auxiliary polynomial becomes:

<u>Aux:</u> $r(r+2)(r^2-7r+12) = r(r+2)(r-3)(r-4) = 0$, which gives you $\boxed{r = 0, -2, 3, 4}$.

And therefore, the general solution to our differential equation is:

$$y(t) = Ae^{0t} + Be^{-2t} + Ce^{3t} + De^{4t} = A + Be^{-2t} + Ce^{3t} + De^{4t}$$

Now notice that $\lim_{t\to\infty} e^{3t} = \infty$, and same for e^{4t}, so if $C \neq 0$ or $D \neq 0$, we get $\lim_{t\to\infty} y(t) = \infty$, and therefore we must have $C = 0$ and $D = 0$.

Therefore $y(t) = A + Be^{-2t}$.

And now we have $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} A + Be^{-2t} = A + 0 = A = 54$, so $A = 54$, and finally we get:

$$y(t) = 54 + Be^{-2t}$$

(where B is any real number). And this indeed satisfies that $\lim_{t\to\infty} y(t) = 54$.

Example: Suppose that we happen to know that $x, x\ln(x), x(\ln(x))^2$ solves:

$$x^3 y''' + xy' - y = 0$$

Find the general solution to this differential equation.

Since we're dealing with a **third**-order differential equation, we have the usual fact:

FACT: The vector space of general solutions to this differential equation is 3-dimensional.

So all we really need to show is that the three functions above are linearly independent (because then we would get a set of 3 linearly independent solutions, and hence a basis for the solution set).

Before you get *too* excited and use the Wronskian, let's be clever about this and simplify our work (this is important to remember when you have complicated functions):

Suppose $ax + bx\ln(x) + cx\left(\ln(x)\right)^2 = 0$.

Then, canceling out x (**provided** $x \neq 0$, because you don't want to divide by 0), we get $a(1) + b\ln(x) + c\left(\ln(x)\right)^2 = 0$.

So all we need to show is that $1, \ln(x), \left(\ln(x)\right)^2$ are linearly independent. **NOW** use the Wronskian:

$$\widetilde{W}(x) = \begin{bmatrix} 1 & \ln(x) & \left(\ln(x)\right)^2 \\ 0 & \frac{1}{x} & 2\frac{\ln(x)}{x} \\ 0 & -\frac{1}{x^2} & 2\left(\frac{1-\ln(x)}{x^2}\right) \end{bmatrix}$$

Now pick a point $x \neq 0$ where the determinant is nonzero. For example $x = 1$ works:

$$\widetilde{W}(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

So $W(1) = 2 \neq 0$, so your three functions are linearly independent.

Therefore $\left\{x, x\ln(x), x\left(\ln(x)\right)^2\right\}$ is a fundamental solution set of your differential equation (= a *linearly independent* set of *three* functions which *solve* your differential equation), and therefore by the **FACT**, the general solution to our differential equation is:

$$y(t) = Ax + Bx\ln(x) + Cx\left(\ln(x)\right)^2$$

Example: Find the largest interval on which:

$$(\tan(t))\, y'' + (t-1)y' + 3y = \tan^2(t)$$

with $y\left(\frac{1}{2}\right) = 0, y'\left(\frac{1}{2}\right) = 1$ has a unique (twice-differentiable) solution.

First, we want to convert the equation into 'standard form' (meaning that the coefficient of y'' has to be 1). So dividing the equation by $\tan(t)$ and using $\tan^2(t) = (\tan(t))^2$, we get:

$$y'' + \left(\frac{t-1}{\tan(t)}\right) + \left(\frac{3}{\tan(t)}\right) y = \tan(t)$$

Now let's find the sets where each terms are continuous (which in this case is the domain of each set).

IMPORTANT: Do **NOT** find the whole domains, just focus on what's happening *near* $\frac{1}{2}$ (our initial condition).

y''-term: $\text{Dom}(1) = \mathbb{R}$

y'-term: $\text{Dom}\left(\frac{t-1}{\tan(t)}\right) = \left(0, \frac{\pi}{2}\right)$ (again, just focus on what's happening near $\frac{1}{2}$. Notice that $\tan(t) = 0$ at $t = 0, \pi$, and $\tan(t)$ isn't defined at $t = \frac{\pi}{2}$. We do not care what's happening at $\frac{3\pi}{2}$ for example!)

y-term: $\text{Dom}\left(\frac{3}{\tan(t)}\right) = \left(0, \frac{\pi}{2}\right)$

Inhomogeneous term: Dom$\tan(t)$ = $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Therefore, the answer we're looking for is the intersection of all those intervals:

$$\mathbb{R} \cap \left(0, \frac{\pi}{2}\right) \cap \left(0, \frac{\pi}{2}\right) \cap \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = \left(0, \frac{\pi}{2}\right)$$

Example: Find the form of a particular solution y_p to:

$$y'' - 2y' + y = f(t)$$

where$\cdots$

$\cdots$ I'm gonna stop here for a second!

Before you even look what f is, find the general solution to the homogeneous equation $y'' - 2y' + y = 0$.

Aux: $r^2 - 2r + 1 = (r-1)^2 = 0 \Rightarrow r = 1$ (double root), and hence:

$$y_0(t) = Ae^t + Bte^t$$

We'll need this to compare the particular solution with the homogeneous solution.

(a) $f(t) = e^t$

First guess $y_p = e^t$, but this coincides with the homogeneous solution. Then guess $y_p = te^t$, ditto. Finally guess $y_p = t^2e^t$. This works, so guess:

$$y_p(t) = At^2e^t$$

(b) $f(t) = te^t$

Always treat the polynomial term separately!

So t becomes $At + B$.

For e^t, guess e^t, which coincides with the homogeneous solution. Then guess te^t, ditto. Finally guess t^2e^t. Bingo.

Therefore, multiplying both things, we get:

$$y_p(t) = (At + B)t^2e^t$$

(c) $f(t) = t^3e^t$

t^3 becomes $At^3 + Bt^2 + Ct + D$

For e^t: guess e^t. No. Guess te^t. No. Guess t^2e^t. Bingo.

So multiplying, we get:

$$y_p(t) = (At^3 + Bt^2 + Ct + D)t^2e^t$$

(d) $f(t) = e^t\cos(t)$

It's *really* important **not** to separate $e^t\cos(t)$ (they're like bread and butter), so here you'd guess:

$$y_p(t) = Ae^t \cos(t) + Be^t \sin(t)$$

(Notice that y_p does *not* coincide with the homogeneous solution)

(e) $f(t) = te^t \cos(t)$

Again, treat the polynomial term separately from the rest:
t becomes $At + B$

$e^t \cos(t)$ becomes $Ae^t \cos(t) + Be^t \sin(t)$

Multiplying everything together (and make sure to label each constant differently:), we get:

$$y_p(t) = (At + B)e^t \cos(t) + (A't + B')e^t \sin(t)$$

3. Systems of Differential Equations

Since this is fresh in your mind, let's just do one example.

Example:

(a) Find the general solution of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 0 & -1 \\ 5 & -2 \end{bmatrix}$.

Eigenvalues: The eigenvalues are $\lambda = -2 \pm i$

(**NOTE:** Someone pointed out afterwards that the eigenvalues are actually $\lambda = -1 \pm 2i$. For the rest of the problem, just do the problem assuming that the eigenvalues are $\lambda = -2 \pm i$ and the eigenvectors are as below. My bad!)

Eigenvectors: For the eigenvalue $\lambda = -2 + i$, the corresponding eigenspace is Span$\left\{ \begin{bmatrix} i \\ 5 \end{bmatrix} \right\}$, which means that the solution is:

$$
\begin{aligned}
\mathbf{x}(t) &= \left(e^{-2t}\cos(t) + ie^{-2t}\sin(t)\right)\left(\begin{bmatrix}0\\5\end{bmatrix} + i\begin{bmatrix}1\\0\end{bmatrix}\right) \\
&= \left(e^{-2t}\cos(t)\begin{bmatrix}0\\5\end{bmatrix} - e^{-2t}\sin(t)\begin{bmatrix}1\\0\end{bmatrix}\right) + i\left(e^{-2t}\cos(t)\begin{bmatrix}1\\0\end{bmatrix} + e^{-2t}\sin(t)\begin{bmatrix}0\\5\end{bmatrix}\right) \\
`&=' A\left(e^{-2t}\cos(t)\begin{bmatrix}0\\5\end{bmatrix} - e^{-2t}\sin(t)\begin{bmatrix}1\\0\end{bmatrix}\right) + B\left(e^{-2t}\cos(t)\begin{bmatrix}1\\0\end{bmatrix} + e^{-2t}\sin(t)\begin{bmatrix}0\\5\end{bmatrix}\right) \\
&= A\begin{bmatrix}-e^{-2t}\sin(t)\\5e^{-2t}\cos(t)\end{bmatrix} + B\begin{bmatrix}e^{-2t}\cos(t)\\5e^{-2t}\sin(t)\end{bmatrix}
\end{aligned}
$$

(b) Find the solution $\mathbf{x}(t)$ which satisfies $\mathbf{x}(0) = \begin{bmatrix}1\\2\end{bmatrix}$

Using the above formula for $\mathbf{x}(t)$, we get:

$$
\mathbf{x}(0) = \begin{bmatrix}B\\5A\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}
$$

And therefore $B = 1$ and $A = \frac{2}{5}$, and therefore:

$$
\mathbf{x}(t) = \frac{2}{5}\begin{bmatrix}-e^{-2t}\sin(t)\\5e^{-2t}\cos(t)\end{bmatrix} + \begin{bmatrix}e^{-2t}\cos(t)\\5e^{-2t}\sin(t)\end{bmatrix}
$$

(c) For $\mathbf{x}(t)$ as in (b), what is $\lim_{t\to\infty}\mathbf{x}(t)$?

Notice $\lim_{t\to\infty} -e^{-2t}\sin(t) = 0$ (by the squeeze theorem), and similarly for the other terms), therefore $\lim_{t\to\infty}\mathbf{x}(t) = \begin{bmatrix}0\\0\end{bmatrix}$

(d) Draw the phase portrait for $\mathbf{x}(t)$ as in (b).

Just like today at noon, because $\mathbf{x}(t)$ involves trig-terms, $\mathbf{x}(t)$ has a spiraly-feature. Moreover by (c), we know that $\mathbf{x}(t)$ has to go to $\begin{bmatrix}0\\0\end{bmatrix}$, and so it follows that $\mathbf{x}(t)$ has to spiral inwards to $\begin{bmatrix}0\\0\end{bmatrix}$, as in the following picture (this is just a technicality, but note that the spiral is slightly elongated along the $y-$ axis because the eigenvector is $\begin{bmatrix}0\\5\end{bmatrix} + i\begin{bmatrix}1\\0\end{bmatrix}$):

54/Math 54 - Fall 2014/Spiral.png