Here are a couple of hints to Homework 26. Enjoy!

Warning: This homework is also very long and very hard! I strongly urge you to read my PDEs-handout before tackling this problem set!

SECTION 10.4: FOURIER COSINE AND SINE SERIES

IMPORTANT NOTE: The book uses the following trick A LOT:

Namely, suppose that when you calculate your coefficients $A_m$ or $B_m$, you get something like: $A_m = \frac{(-1)^{m+1} + 1}{\pi m}$.

Then notice the following: If $m$ is even, then $(-1)^{m+1} + 1 = 0$, so $A_m = 0$, and if $m$ is odd, $(-1)^{m+1} + 1 = -2$, and $A_m = \frac{-2}{\pi m}$.

So at some point, you would like to say:

$$f(x) = \sum_{m=1, \text{odd}}^{\infty} A_m \cos(mx)$$

The way you do this is as follows: Since $m$ is odd $m = 2k - 1$, for $k = 1, 2, 3 \cdots$, and so the sum becomes:

$$f(x) = \sum_{k=1}^{\infty} \frac{-2}{\pi(2k - 1)} \cos((2k - 1)x)$$

Date: Monday, December 15th, 2014.
10.4.1, 10.4.3. \( \pi \)-periodic extension just means ‘repeat the graph of \( f \)’.

The even-\( 2\pi \) periodic extension is just the function:

\[
f_e(x) = \begin{cases} 
  f(-x) & \text{if } -\pi < x < 0 \\
  f(x) & \text{if } 0 < x < \pi
\end{cases}
\]

The odd-\( 2\pi \) periodic extension is just the function:

\[
f_o(x) = \begin{cases} 
  -f(-x) & \text{if } -\pi < x < 0 \\
  0 & \text{if } x = 0 \\
  f(x) & \text{if } 0 < x < \pi
\end{cases}
\]

And repeat all those graphs!

10.4.5, 10.4.7, 10.4.9. Use the formulas:

\[
f(x) = \sum_{m=0}^{\infty} A_m \cos \left( \frac{\pi mx}{T} \right)
\]

where:

\[
A_0 = \frac{1}{T} \int_{0}^{T} f(x) \, dx
\]

\[
A_m = \frac{2}{T} \int_{0}^{T} f(x) \cos \left( \frac{\pi mx}{T} \right) \, dx
\]

10.4.11, 10.4.13. Use the formulas:

\[
f(x) = \sum_{m=0}^{\infty} B_m \sin \left( \frac{\pi mx}{T} \right)
\]

where:

\[
B_0 = 0
\]

\[
B_m = \frac{2}{T} \int_{0}^{T} f(x) \sin \left( \frac{\pi mx}{T} \right) \, dx
\]

10.4.17, 10.4.19. See next section!
The best advice I can give you is: Read the PDE handout, specifically the section about the heat equation! It outlines all the important steps you’ll need!

Also read the **important note** I wrote in the previous section!

**10.5.7.** Imitate Example 2! Your solution is

\[ u(x, t) = v(x) + w(x, t) \]

where \( v(x) = 5 + \frac{5x}{\pi} \) and \( w(x, t) \) solves the corresponding homogeneous equation with \( w(0, t) = 0, w(\pi, t) = 0 \) but with \( w(x, 0) = \sin(3x) - \sin(5x) - v(x) \).

**10.5.9.** Don’t worry about this for the exam, but basically because we’re dealing with an inhomogeneous solution, the general solution \( u(x, t) \) is of the following form:

\[ u(x, t) = v(x) + w(x, t) \]

where \( v(x) \) is a **particular** solution to the differential equation, and \( w(x, t) \) is the general solution to the **homogeneous** equation (36), (37), (38) on page 671 (careful about the initial term, it’s \( w(x, 0) = f(x) - v(x) \), not \( w(x, 0) = f(x) \)).

To find \( v \) use formula (35) on page 671, and to find \( w \), solve equations (36), (37), (38).

**10.5.15, 10.5.17.** Note: This is just an outline. On your homework, please fill in all the details.

This time assume \( u(x, y, t) = X(x)Y(y)T(t) \). Plugging \( u \) into the PDE we get:

\[ X(x)Y(y)T'(t) = X''(x)Y(y)T(t) + XY''(y)T(t) \]

And dividing by \( X(x)Y(t)T(t) \), we get:

\[ \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} \]
Now the right-hand-side depends only on $x$, and $y$, but also only on $t$ (by the left-hand-side), hence it is constant, which gives us:

$$\frac{T'(t)}{T(t)} = \lambda = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}$$

So in particular $T'(t) = \lambda T(t)$, and also:

$$\frac{X''(x)}{X(x)} = \lambda - \frac{Y''(y)}{Y(y)}$$

But now notice that the left-hand-side depends only on $x$, and only on $y$ (by the right-hand-side), hence it is also constant, and we get:

$$\frac{X''(x)}{X(x)} = \mu = \lambda - \frac{Y''(y)}{Y(y)}$$

So in particular $X''(x) = \mu X(x)$. Now use the boundary condition $X'(0) = X'(\pi) = 0$ and cases to argue that $\mu = m^2$ for $m = 0, 1, \cdots$, and $X_m = A_m \cos(mx)$ which gives:

$$\frac{Y''(y)}{Y(y)} = \lambda - \mu = \lambda - m^2$$

But now use the boundary condition $Y(0) = Y(\pi) = 0$ and cases to argue that $\lambda - m^2 = n^2$ for $n = 1, 2, 3, \cdots$, (and so $\lambda = m^2 + n^2$), and $Y_n = B_n \sin(ny)$.

Finally, using $T'(t) = \lambda T = (m^2 + n^2)T$, we get: $T(t) = e^{(m^2+n^2)t}$, and we finally obtain:

$$u_{mn}(x, y, t) = X(x)Y(y)T(t) = A_mB_n \cos(mx) \sin(ny)e^{(m^2+n^2)t}$$

And finally the general solution is:

$$u(x, y, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} C_{mn} \cos(mx) \sin(ny)e^{(m^2+n^2)t}$$

Finally, all you have to do is to plug in $t = 0$ to get:

$$u(x, y, 0) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} C_{mn} \cos(mn) \sin(ny)$$

For 10.5.15, you just have to compare terms and get $C_{6,4} = 1$ and $C_{1,11} = -3$ and everything else $= 0$, which tells you:

$$u(x, y, t) = \cos(6x) \sin(4y)e^{52t} - 3 \cos(x) \sin(11y)e^{122t}$$
And for 10.5.17, you use 'hugging' (or orthogonality) to get:

\[
C_{0n} = \frac{\int_0^\pi \int_0^\pi y(1) \sin(ny) \, dx \, dy}{\int_0^\pi \int_0^\pi 1^2 \sin^2(ny) \, dx \, dy} = \frac{\pi \int_0^\pi y \sin(ny) \, dy}{\pi} = \frac{2(-1)^{n+1}}{n}
\]

If \( m \geq 1 \)

\[
C_{mn} = \frac{\int_0^\pi \int_0^\pi y \cos(mx) \sin(ny) \, dx \, dy}{\int_0^\pi \int_0^\pi \cos^2(mx) \sin^2(ny) \, dx \, dy} = \frac{(\int_0^\pi \cos(mx) \, dx) (\int_0^\pi y \sin(ny) \, dy)}{\frac{1}{4}} = 0
\]

Which ultimately gives us:

\[
u(x, y, t) = \sum_{n=1}^{\infty} 2 \frac{(-1)^{n+1}}{n} e^{-n^2t} \sin(ny)
\]