MATH 54 – SOLUTION TO 10.2.21

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Step 1: Separation of variables. Suppose:

$$u(x,t) = X(x)T(t) \tag{1}$$

Plug (1) into the differential equation we get:

$$\begin{aligned}(X(x)T(t))_{tt} =& 9\,(X(x)T(t))_{xx}\\ X(x)T''(t) =& 9X''(x)T(t)\end{aligned}$$

Rearrange and get:

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{9T(t)} \tag{2}$$

Now $\frac{X''(x)}{X(x)}$ *only* depends on x, but by (2) *only* depends on t, hence it is constant:

$$\begin{aligned}\frac{X''(x)}{X(x)} =& \lambda\\ X''(x) =& \lambda X(x)\end{aligned} \tag{3}$$

Also, we get:

$$\begin{aligned}\frac{T''(t)}{9T(t)} =& \lambda\\ T''(t) =& 9\lambda T(t)\end{aligned} \tag{4}$$

Date: Monday, December 8, 2014.

Step 2: Consider (3):

$$X''(x) = \lambda X(x)$$

Now use the **boundary conditions**:

$$u(0,t) = X(0)T(t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0$$

$$u(\pi,t) = X(\pi)T(t) = 0 \Rightarrow X(\pi)T(t) = 0 \Rightarrow X(\pi) = 0$$

Hence we get:

$$\begin{cases} X''(x) = \lambda X(x) \\ X(0) = 0 \\ X(\pi) = 0 \end{cases} \tag{5}$$

Step 3: Eigenvalues/Eigenfunctions. The auxiliary polynomial of (5) is $p(\lambda) = r^2 - \lambda$

Now we need to consider 3 cases:

<u>Case 1:</u> $\lambda > 0$, then $\lambda = \omega^2$, where $\omega > 0$

Then:

$$r^2 - \lambda = 0 \Rightarrow r^2 - \omega^2 = 0 \Rightarrow r = \pm\omega$$

Therefore:

$$X(x) = Ae^{\omega x} + Be^{-\omega x}$$

Now use $X(0) = 0$ and $X(\pi) = 0$:

$$X(0) = 0 \Rightarrow A + B = 0 \Rightarrow B = -A \Rightarrow X(x) = Ae^{\omega x} - Ae^{-\omega x}$$

$$X(\pi) = 0 \Rightarrow Ae^{\omega\pi} - Ae^{-\omega\pi} = 0 \Rightarrow Ae^{\omega\pi} = Ae^{-\omega\pi} \Rightarrow e^{\omega\pi} = e^{-\omega\pi} \Rightarrow \omega\pi = -\omega\pi \Rightarrow \omega = 0$$

But this is a **contradiction**, as we want $\omega > 0$.

<u>Case 2:</u> $\lambda = 0$, then $r = 0$, and:

$$X(x) = Ae^{0x} + Bxe^{0x} = A + Bx$$

And:

$$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = Bx$$

$$X(\pi) = 0 \Rightarrow B = 0 \Rightarrow X(x) = 0$$

Again, a **contradiction** (we want $X \not\equiv 0$, because otherwise $u(x,t) \equiv 0$)

<u>Case 3:</u> $\lambda < 0$, then $\lambda = -\omega^2$, and:

$$r^2 - \lambda = 0 \Rightarrow r^2 + \omega^2 = 0 \Rightarrow r = \pm \omega i$$

Which gives:

$$X(x) = A\cos(\omega x) + B\sin(\omega x)$$

Again, using $X(0) = 0$, $X(\pi) = 0$, we get:

$$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = B\sin(\omega x)$$

$$X(\pi) = 0 \Rightarrow B\sin(\omega\pi) = 0 \Rightarrow \sin(\omega\pi) = 0 \Rightarrow \omega = m, \quad (m = 1, 2, \cdots)$$

This tells us that:

(6)
$$\begin{aligned} \text{Eigenvalues:} \lambda &= -\omega^2 = -m^2 \quad (m = 1, 2, \cdots) \\ \text{Eigenfunctions:} X(x) &= \sin(\omega x) = \sin(mx) \end{aligned}$$

Step 4: Deal with (4), and remember that $\lambda = -m^2$:

$$T''(t) = 3\lambda T(t)$$

<u>Aux:</u> $r^2 = -9m^2 \Rightarrow r = \pm 3mi \qquad (m = 1, 2, \cdots)$

$$T(t) = A_m \cos(3mt) + B_m \sin(3mt)$$

Step 5: Take linear combinations:

(7)
$$u(x,t) = \sum_{m=1}^{\infty} T(t)X(x) = \sum_{m=1}^{\infty} \left(\widetilde{A_m}\cos(3mt) + \widetilde{B_m}\sin(3mt)\right)\sin(mx)$$

Step 6: Use the initial condition $u(x,0) = 6\sin(2x) + 2\sin(6x)$:

Plug in $t = 0$ in (7), and you get:

$$u(x,0) = \sum_{m=1}^{\infty} A_m \sin(mx) = 6\sin(2x) + 2\sin(6x) \qquad \text{on}(0,\pi) \tag{8}$$

Equating coefficients, you get:

$$\begin{aligned} A_2 &= 6 && \text{(coefficient of } \sin(2x)) \\ A_6 &= 2 && \text{(coefficient of } \sin(6x)) \\ A_m &= 0 && \text{(for all other } m) \end{aligned}$$

Step 7: Use the initial condition: $\frac{\partial u}{\partial t}(x,0) = 11\sin(9x) - 14\sin(15x)$:
First differentiate (7) with respect to t:

$$\frac{\partial u}{\partial t}(x,t) = \sum_{m=1}^{\infty} \left(-3mA_m \sin(3mt) + 3mB_m \cos(3mt)\right)\sin(mx) \tag{9}$$

Now plug in $t = 0$ in (9):

$$\frac{\partial u}{\partial t}(x,0) = \sum_{m=1}^{\infty} 3m\widetilde{B_m} \sin(mx) = 11\sin(9x) - 14\sin(15x) \tag{10}$$

Equating coefficients, you get:

$$\begin{aligned} 27B_9 &= 11 && \text{(coefficient of } \sin(9x)) \\ 45B_{15} &= -14 && \text{(coefficient of } \sin(15x)) \\ B_m &= 0 && \text{(for all other } m) \end{aligned}$$

That is:

$$\begin{aligned} B_9 &= \frac{11}{27} && \text{(coefficient of } \sin(9x)) \\ B_{15} &= -\frac{14}{45} && \text{(coefficient of } \sin(15x)) \\ B_m &= 0 && \text{(for all other } m) \end{aligned}$$

Step 8: Conclude using (7) and the coefficients A_m and B_m you found:

$$u(x,t) = \sum_{m=1}^{\infty} \left(A_m \cos(3mt) + B_m \sin(3mt)\right) \sin(mx) \tag{11}$$

where:

$$\begin{aligned} A_2 &= 6 \\ A_6 &= 2 \\ A_m &= 0 \qquad \text{(for all other } m\text{)} \end{aligned}$$

and

$$\begin{aligned} B_9 &= \frac{11}{27} \\ B_{15} &= -\frac{14}{45} \\ B_m &= 0 \qquad \text{(for all other } m\text{)} \end{aligned}$$

Note: In this *special* case, you can write $u(x,t)$ in the following nice form:

(12)

$$u(x,t) = 6\cos(6t)\sin(2x) + 2\cos(18t)\sin(6x) + \frac{11}{27}\sin(27t)\sin(9x) - \frac{14}{45}\sin(45t)\sin(15x)$$