MATH 54 – SOLUTION TO 6.2.20

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General solution:

The auxiliary equation is $p(r) = r^3 + 7r^2 + 14r + 8 = 0$.

To factor this, we use the rational roots theorem, which says that if r is a root of the form $\frac{a}{b}$, then a divides 8 (so $a = \pm 1, \pm 2, \pm 4, \pm 8$), and b divides 1 (so $b = \pm 1$). This gives us the guesses $r = \pm 1, \pm 2, \pm 4, \pm 8$:

$$p(1) = 1 + 7 + 14 + 8 = 30 \neq 0$$

$$p(-1) = -1 + 7 - 14 + 8 = 0$$

BINGO! Therefore $r = -1$ is a root, and to factor out p, we use long-division:

$$\begin{array}{r|l} & X^2 + 6X + 8 \\ \hline X+1 & X^3 + 7X^2 + 14X + 8 \\ & -X^3 - X^2 \\ \hline & 6X^2 + 14X \\ & -6X^2 - 6X \\ \hline & 8X + 8 \\ & -8X - 8 \\ \hline & 0 \end{array}$$

Therefore $p(r) = (r+1)(r^2 + 6r + 8) = 0$, which gives us $r = -1$, or $r^2 + 6r + 8 = 0$, that is:

$$r = \frac{-6 \pm \sqrt{36 - 32}}{2} = \frac{-6 \pm 2}{2} = -4, -2$$

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And therefore the roots of p are $\boxed{r = -1, -2, -4}$, which tells us that the general solution to our differential equation is:

$$y(t) = Ae^{-t} + Be^{-2t} + Ce^{-4t}$$

<u>Initial conditions:</u>

Plugging in $t = 0$, we get $y(0) = A + B + C = 1$

Differentiating, we get:

$$y'(t) = -Ae^{-t} - 2Be^{-2t} - 4Ce^{-4t}$$

Plugging in $t = 0$, we get $y'(0) = -A - 2B - 4C = -3$

Differentiating again, we get:

$$y''(t) = Ae^{-t} + 4Be^{-2t} + 16Ce^{-4t}$$

Plugging in $t = 0$, we get $y''(0) = A + 4B + 16C = 13$.

So we are led to solve the system:

$$\begin{cases} A + B + C = 1 \\ -A - 2B - 4C = -3 \\ A + 4B + 16C = 13 \end{cases}$$

Which we can solve by row-reduction (yay!!!):

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -2 & -4 & -3 \\ 1 & 4 & 16 & 13 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

which gives us $\boxed{A = 1, B = -1, C = 1}$, and therefore the solution of our differential equation is:

$$y(t) = e^{-t} - e^{-2t} + e^{-4t}$$