MATH 54 – SOLUTION TO 4.5.21

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Homogeneous equation:

First of all, the auxiliary equation is $r^2 + 2r + 2 = 0$, which gives:

$$r = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

which tells you the general solution of $y'' + 2y' + 2 = 0$ is:

$$y_0(\theta) = Ae^{-\theta}\cos(\theta) + Be^{-\theta}\sin(\theta)$$

Particular solution:

Notice that $e^{-\theta}\cos(\theta)$ is *already* a solution of the homogeneous equation, so we'll have to guess:

$$y_p(\theta) = A\theta e^{-\theta}\cos(\theta) + B\theta e^{-\theta}\sin(\theta)$$

This gives us:

$$\begin{aligned} y_p'(\theta) =& Ae^{-\theta}\cos(\theta) - A\theta e^{-\theta}\cos(\theta) - A\theta e^{-\theta}\sin(\theta) + Be^{-\theta}\sin(\theta) - B\theta e^{-\theta}\sin(\theta) + B\theta e^{-\theta}\cos(\theta) \\ =& (A - A\theta + B\theta)e^{-\theta}\cos(\theta) + (B - B\theta - A\theta)e^{-\theta}\sin(\theta) \end{aligned}$$

And:

$$\begin{aligned} y_p''(\theta) =& (B-A)e^{-\theta}\cos(\theta) - (A - A\theta + B\theta)e^{-\theta}\cos(\theta) - (A - A\theta + B\theta)e^{-\theta}\sin(\theta) \\ &+ (-B-A)e^{-\theta}\sin(\theta) - (B - B\theta - A\theta)e^{-\theta}\sin(\theta) + (B - B\theta - A\theta)e^{-\theta}\cos(\theta) \\ =& (B - A - A + \cancel{A\theta} - B\theta + B - B\theta - \cancel{A\theta})e^{-\theta}\cos(\theta) \\ &+ (-A + A\theta - \cancel{B\theta} - B - A - B + \cancel{B\theta} + A\theta)e^{-\theta}\sin(\theta) \\ =& (2B - 2A - 2B\theta)e^{-\theta}\cos(\theta) + (-2A - 2B + 2A\theta)e^{-\theta}\sin(\theta) \end{aligned}$$

Plugging those formulas into our equation $y'' + 2y' + 2y = e^{-\theta}\cos(\theta)$, we get:

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$$\begin{aligned} y_p'' + 2y_p' + 2y_p &= e^{-\theta}\cos(\theta) \\ \left[(2B - 2A - 2B\theta)e^{-\theta}\cos(\theta) + (-2A - 2B + 2A\theta)e^{-\theta}\sin(\theta)\right] + \\ 2\left[(A - A\theta + B\theta)e^{-\theta}\cos(\theta) + (B - B\theta - A\theta)e^{-\theta}\sin(\theta)\right] \\ +2\left[A\theta e^{-\theta}\cos(\theta) + B\theta e^{-\theta}\sin(\theta)\right] &= e^{-\theta}\cos(\theta) \\ \left(2B - \cancel{2A} - \cancel{2B\theta} + \cancel{2A} - \cancel{2A\theta} + \cancel{2B\theta} + \cancel{2A\theta}\right) e^{-\theta}\cos(\theta) \\ + \left(-2A - \cancel{2B} + \cancel{2A\theta} + \cancel{2B} - \cancel{2B\theta} - \cancel{2A\theta} + \cancel{2Bt}\right) e^{-\theta}\sin(\theta) \\ &= e^{-\theta}\cos(\theta) \\ 2Be^{-\theta}\cos(\theta) + (-2A)e^{-\theta}\sin(\theta) &= 1e^{-\theta}\cos(\theta) + 0e^{-\theta}\sin(\theta) \end{aligned}$$

Comparing the left-hand-side and the right-hand-side, we get $2B = 1$ and $-2A = 0$, so $\boxed{A = 0}$ and $\boxed{B = \frac{1}{2}}$, which tells us that a particular solution is:

$$y_p(\theta) = 0\theta e^{-\theta}\cos(\theta) + \frac{1}{2}\theta e^{-\theta}\sin(\theta) = \frac{1}{2}\theta e^{-\theta}\sin(\theta)$$

<u>General solution:</u> And therefore the general solution to our differential equation is:

$$y(\theta) = y_0(\theta) + y_p(\theta) = Ae^{-\theta}\cos(\theta) + Be^{-\theta}\sin(\theta) + \frac{1}{2}\theta e^{-\theta}\sin(\theta)$$