Final Exam – Review – Part I

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Monday, December 8, 2014

This review focuses on the linear algebra material from after midterm 2. Since the final exam is cumulative, make sure to look over the review problems from Midterm 1 and Midterm 2 as well.

1 Symmetric matrices

Problem 1

Find a diagonal matrix D and an orthogonal matrix P such that $A = PDP^T$, where

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

2 Least squares

Problem 2

Find the least squares solution and the least-squares error to $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

Problem 3

Let:

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

(a) Find the $QR-$decomposition of A. (apparently this is fair game for the exam)

(b) Find the orthogonal projection of $\mathbf{b}$ on $Col(A)$

(c) Use (b) to find the least-squares solution of $A\mathbf{x} = \mathbf{b}$

(d) Use $A = QR$ to find the least-squares solution of $A\mathbf{x} = \mathbf{b}$

3 Inner product spaces

Problem 4

Find the orthogonal projection of $f(x) = x$ on W and use this to find a function $g(x)$ orthogonal to W, where:

$$W = Span\left\{\sin(x), \cos(x), \cos(2x)\right\}$$

with respect to the inner product:

$$f \cdot g = \int_{-\pi}^{\pi} f(x)g(x)dx$$

4 True/False Extravaganza

Problem 5

(a) Any two linearly independent eigenvectors of a symmetric matrix are orthogonal

(b) If Q is orthogonal, then $QQ^T = I$

(c) The Gram-Schmidt process applied to the columns of a matrix A preserves the column space of A

(d) If A is orthogonal, then $Row(A) = Col(A)$

(e) For any functions f and g, we have:

$$\int_0^1 f(x)g(x)dx \leq \left(\int_0^1 (f(x))^2\, dx\right)^{\frac{1}{2}} \left(\int_0^1 (g(x))^2\, dx\right)^{\frac{1}{2}}$$

(f) The equation $A\mathbf{x} = \mathbf{b}$ always has a least-squares solution.

(g) The equation $A\mathbf{x} = \mathbf{b}$ cannot have more than one least-squares solution.