Final Exam – Review – Answers

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1 Part I

1.
$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

2. In the original problem, the 1 in $\mathbf{b}$ should be an 11.

$$\widetilde{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\|A\widetilde{\mathbf{x}} - \mathbf{b}\| = \sqrt{84} = 2\sqrt{21}$$

3. (a) $A = QR$, where:

$$Q = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}, \quad R = \begin{bmatrix} 3 & 5 \\ 0 & -1 \end{bmatrix}$$

(b) $\hat{\mathbf{b}} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$

(c) $\widetilde{\mathbf{x}} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ (solve $A\widetilde{\mathbf{x}} = \mathbf{b}$)

(d) $\widetilde{\mathbf{x}} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ (solve $R\widetilde{\mathbf{x}} = Q^T\mathbf{b}$)

4. $\hat{f}(x) = 2\sin(x)$, $g(x) = f(x) - \hat{f}(x) = x - 2\sin(x)$

5. Since I didn't have time to go over this during the review session, you can find detailed explanations at the end of this document.

   (a) $\boxed{\textbf{FALSE}}$

   (b) $\boxed{\textbf{TRUE}}$

   (c) $\boxed{\textbf{TRUE}}$

   (d) $\boxed{\textbf{TRUE}}$

   (e) $\boxed{\textbf{TRUE}}$

   (f) $\boxed{\textbf{TRUE}}$

   (g) $\boxed{\textbf{FALSE}}$

2 Part II

1. See Midterm 2 - review for the eigenvalues/eigenvectors

$$\begin{aligned}\mathbf{x}(t) = Ae^{3t}\begin{bmatrix}1\\1\\1\end{bmatrix} + B\left(e^t\cos(2t)\begin{bmatrix}-1\\1\\2\end{bmatrix} - e^t\sin(2t)\begin{bmatrix}-1\\1\\0\end{bmatrix}\right)\\ + C\left(e^t\cos(2t)\begin{bmatrix}-1\\1\\0\end{bmatrix} + e^t\sin(2t)\begin{bmatrix}-1\\1\\2\end{bmatrix}\right)\end{aligned}$$

2. $$\mathbf{x}(t) = Ae^{2t}\begin{bmatrix}1\\1\end{bmatrix} + B\left(te^{2t}\begin{bmatrix}1\\1\end{bmatrix} + e^{2t}\begin{bmatrix}0\\-1/3\end{bmatrix}\right)$$

3. $(0, 2)$

4. $y(x) = Ax\sin(x)e^x + B\ln(x)e^x$ (just show your functions are linearly independent by canceling out the e^x first, and then using the Wronskian at $x = \pi$)

5. $y(t) = Ae^{-1/2t} + Be^{-t}\cos(2t) + Ce^{-t}\sin(2t)$, and use the squeeze theorem to show $\lim_{t\to\infty} y(t) = 0$

6. First of all, the solution to the homogeneous equation is:

$$\begin{aligned} y_0(t) = A + Bt + Ct^2 + De^t + Ete^t + Ft^2e^t + \\ G\cos(2t) + H\sin(2t) + It\cos(2t) + Jt\sin(2t) + \\ Ke^t\cos(2t) + Le^t\sin(2t) + Mte^t\cos(2t) + Nte^t\sin(2t) \end{aligned}$$

   (a) $y(t) = (At^2 + Bt + C)t^3e^t$
   (b) $y(t) = (At + B)t^2e^t\cos(2t) + (Ct + D)t^2e^t\sin(2t)$
   (c) $y(t) = Ae^t\cos(3t) + Be^t\sin(3t)$
   (d) $y(t) = Ae^{2t}\cos(2t) + Be^{2t}\sin(2t)$
   (e) $y(t) = (At^2 + Bt + C)t^2\cos(2t) + (Dt^2 + Et + F)t^2\sin(2t)$

3 Part III

$$u(x,t) = A_0 + B_0\,t + \sum_{m=1}^{\infty} \left(A_m\cos(2\pi mt) + B_m\sin(2\pi mt)\right)\cos(\pi mx)$$

where:

$$A_0 = -\frac{7}{6}, \quad A_m = \frac{1}{(\pi m)^2}\left(2(-1)^{m+1} + 6\right) \quad (m \geq 1)$$

$$B_0 = 0, \quad B_1 = \frac{3}{2\pi}, \quad B_2 = \frac{5}{4\pi}, \quad B_m = 0 \quad (m \geq 3)$$

Note: For a complete solution to a similar problem, look at Problem 9 on my mock exam.

4 Appendix: Solutions to the T/F extravaganza

(a) $\boxed{\textbf{FALSE}}$

Eigenvectors corresponding to **different** eigenvalues are orthogonal. For example, in the matrix in Problem 1, two eigenvectors corresponding to $\lambda = 2$ are $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, and they are not orthogonal. This is *precisely* why you have to apply Gram-Schmidt to make them orthogonal.

(b) $\boxed{\textbf{TRUE}}$

Since orthogonal matrices are *square* and $Q^TQ = I$, this implies that Q is invertible and $Q^{-1} = Q^T$. But by definition of invertible matrices, we have $Q^{-1}Q = QQ^{-1} = I$, but since $Q^{-1} = Q^T$, we get $Q^TQ = QQ^T = I$, so in particular $\boxed{QQ^T = I}$.

(c) $\boxed{\textbf{TRUE}}$

This is precisely the point of the Gram-Schmidt process, namely given a set $\mathcal{B}$ of vectors, produce an orthonormal set $\mathcal{B}'$ such that $Span\, \mathcal{B}' = Span\, \mathcal{B}$ (i.e. both sets have the same span). For example, suppose $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$ are the columns of A, and you apply Gram-Schmidt to obtain $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$, then if $A' = \begin{bmatrix} \mathbf{w_1} & \mathbf{w_2} & \mathbf{w_3} \end{bmatrix}$, then $Col(A') = Span\, \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\} = Span\, \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\} = Col(A)$. By the way, this is precisely why in Problem 3b, we have $Col(Q) = Col(A)$ (because you obtain Q by applying Gram-Schmidt to the columns of A)

(d) $\boxed{\textbf{TRUE}}$

If A is orthogonal, then A is invertible, so $Row(A) = \mathbb{R}^n = Col(A)$

(e) $\boxed{\textbf{TRUE}}$

There are really only two inequalities we have learned in this course: The Cauchy-Schwarz inequality and the Triangle inequality.

The Cauchy-Schwarz inequality (which we'll need here) says:

$$f \cdot g \leq \|f\| \, \|g\|$$

Applying this with $f \cdot g = \int_0^1 f(x)g(x)dx$, and:

$$\|f\| = (f \cdot f)^{\frac{1}{2}} = \left(\int_0^1 f(x)f(x)dx\right)^{\frac{1}{2}} = \left(\int_0^1 (f(x))^2 \, dx\right)^{\frac{1}{2}}$$

And similar with g, we obtain in fact that:

$$\int_0^1 f(x)g(x)dx \leq \left(\int_0^1 (f(x))^2 \, dx\right)^{\frac{1}{2}} \left(\int_0^1 (g(x))^2 \, dx\right)^{\frac{1}{2}}$$

(f) $\boxed{\textbf{TRUE}}$

This is precisely because $\hat{\mathbf{b}}$ is in $Col(A)$, and therefore by definition of $Col(A)$, the equation $A\widetilde{\mathbf{x}} = \hat{\mathbf{b}}$ is guaranteed to have at least one solution.

(g) $\boxed{\textbf{FALSE}}$

There could be more than one least-squares solution. For example, take $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $A^T A\widetilde{\mathbf{x}} = A^T\mathbf{b}$ becomes $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \widetilde{\mathbf{x}} = \mathbf{0}$, which has infinitely many solutions. In fact, there is a theorem which says: there is only one least-squares solution if and only if $Rank(A) = n$ (where n is the number of columns of A)