Midterm 2 – Review – Problems

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1 Vector space-stuff

Problem 1

Find $\dim(W)$, where:

$$W = \text{Span}\left\{1 - t + 5t^2, -4 + 2t - 6t^2, 9 - 4t + 10t^2, -7 + t + 7t^2\right\}$$

Problem 2

Consider

$$A = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 1 & 2 & -3 & 0 & -2 & -3 \\ 1 & -1 & 0 & 0 & 1 & 6 \\ 1 & -2 & 2 & 1 & -3 & 0 \\ 1 & -2 & 1 & 0 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 0 & 1 & -1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 1 & -13 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Find $Rank(A)$, $\dim(Col(A))$, $\dim(Row(A))$, $\dim(Nul(A))$

(b) Find a basis for $Row(A)$ and a basis for $Col(A)$

(c) Find a basis for $Nul(A)$

Problem 3

If $\mathcal{B} = \{-1 + 8t, 1 - 5t\}$ and $\mathcal{C} = \{1 + 4t, 1 + t\}$, find $\mathcal{C} \overset{P}{\leftarrow} \mathcal{B}$ and use this to calculate $[p]_{\mathcal{B}}$ gien $p = 1(1 + 4t) + 4(1 + t)$.

2 Diagonalization

Problem 4

(a) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$ where:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -3 & 4 \\ 4 & -8 & 7 \end{bmatrix}$$

(b) With A as above, find a matrix C of the form $C = \begin{bmatrix} c & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{bmatrix}$ with a, b, c **real** and an invertible matrix P with **real** entries such that $A = PCP^{-1}$

Problem 5

Find a nonzero vector $\mathbf{v}$ such that $\lim_{n\to\infty} A^n \mathbf{v} = \mathbf{0}$, where:

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

3 Linear transformations

Problem 6

Let $T : P_2 \longrightarrow M_{2\times 2}$ be the following linear transformation:

$$T(p) = \begin{bmatrix} p(0) & p'(1) \\ p''(2) & p(0) \end{bmatrix}$$

(a) Find the matrix T with respect to the basis $\mathcal{B} = \{1, t, t^2\}$ of P_2 and the basis $\mathcal{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ of $M_{2\times 2}$

(b) Find $Nul(T)$ and $Ran(T)$

Problem 7

Find the $\mathcal{B}-$matrix for $T\mathbf{x} = A\mathbf{x}$, where:

$$A = \begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}, \qquad \mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

4 Orthogonal projections

Problem 8

Consider

$$W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}, \quad \mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

(a) Find the orthogonal projection $\hat{\mathbf{x}}$ of $\mathbf{x}$ on W

(b) Find a vector orthogonal to W

(c) Find the distance between $\mathbf{x}$ and W

(d) The three vectors above form a basis $\mathcal{B}$ for W. Find the coordinates of $\begin{bmatrix} 1 \\ 1 \\ 0 \\ -4 \end{bmatrix}$ with respect to $\mathcal{B}$.

5 True/False Extravaganza!

Problem 9

(a) If A is diagonalizable, then A is invertible

(b) If A is invertible, then A is diagonalizable

(c) If A is 3×3 and is diagonalizable, then A must have 3 distinct eigenvalues

(d) If $A = PDP^{-1}$, then A has the same eigenvalues as D

(e) The set of all eigenvectors of A is a vector space.

(f) If $A^2 = A$, then the only eigenvalues of A are 0 and 1

(g) A matrix with complex eigenvalues is not diagonalizable

(h) If A is diagonalizable and invertible, then A^{-1} is diagonalizable

(i) The columns of the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ are the $\mathcal{B}-$coordinates of the vectors in $\mathcal{C}$

(j) $Rank\,(A^2) = Rank(A)$

(k) $\hat{\mathbf{x}}$ is orthogonal to $\mathbf{x}$

(l) The only vector orthogonal to itself is the zero-vector

(m) The matrix $\begin{bmatrix} 0.8 & 0 & -0.6 \\ 0 & 0.1 & 0 \\ 0.6 & 0 & 0.8 \end{bmatrix}$ is orthogonal

(n) Orthogonal matrices are square

(o) If A (not necessarily square) has orthonormal columns, then $AA^T\mathbf{x} = \hat{\mathbf{x}}$