MATH 54 – MOCK MIDTERM 2 – SOLUTIONS

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1. *(30 points, 5 pts each)*

Label the following statements as **T** or **F**.

Make sure to **JUSTIFY YOUR ANSWERS!!!** You may use any facts from the book or from lecture.

(a) If $\mathcal{A} = \{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$ and $\mathcal{D} = \{\mathbf{d_1}, \mathbf{d_2}, \mathbf{d_3}\}$ are bases for V, and P is the matrix whose ith column is $[\mathbf{d_i}]_{\mathcal{A}}$, then for all $\mathbf{x}$ in V, we have $[\mathbf{x}]_{\mathcal{D}} = P\,[\mathbf{x}]_{\mathcal{A}}$

FALSE

First of all, $P = \begin{bmatrix}[\mathbf{d_1}]_{\mathcal{A}} & [\mathbf{d_2}]_{\mathcal{A}} & [\mathbf{d_3}]_{\mathcal{A}}\end{bmatrix} = \mathcal{A} \overset{P}{\leftarrow} \mathcal{D}$ (remember, you always evaluate with respect to the new, cool basis, here it is $\mathcal{A}$), so we should have:

$$[\mathbf{x}]_{\mathcal{A}} = \mathcal{A} \overset{P}{\leftarrow} \mathcal{D}\,[\mathbf{x}]_{\mathcal{D}} = P\,[\mathbf{x}]_{\mathcal{D}}$$

And not the opposite!

(b) A 3×3 matrix A with only one eigenvalue cannot be diagonalizable

SUPER FALSE!!!!!!!!!!

Remember that to check if a matrix is not diagonalizable, you really have to look at the eigenvectors!

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For example, $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ has only eigenvalue 2, but is diagonalizable (it's diagonal!). Or you can choose A to be the O matrix, or the identity matrix, this also works!

(c) If $\mathbf{v_1}$ and $\mathbf{v_2}$ are 2 eigenvectors of A corresponding to 2 **different** eigenvalues λ_1 and λ_2, then $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent!

TRUE (finally!)

Note: The proof is a bit complicated, but I've seen this on a past exam! I think at that point, the professor wanted to get revenge on his students for not coming to lecture!

Remember that eigenvectors have to be nonzero!

Now, assume $a\mathbf{v_1} + b\mathbf{v_2} = \mathbf{0}$.

Then apply A to this to get:

$$A(a\mathbf{v_1} + b\mathbf{v_2}) = A(\mathbf{0}) = \mathbf{0}$$

That is:

$$aA(\mathbf{v_1}) + bA(\mathbf{v_2}) = \mathbf{0}$$

$$a\lambda_1\mathbf{v_1} + b\lambda_2\mathbf{v_2} = \mathbf{0}$$

However, we can also multiply the original equation by λ_1 to get:

$$a\lambda_1\mathbf{v_1} + b\lambda_1\mathbf{v_2} = \mathbf{0}$$

Subtracting this equation from the one preceding it, we get:

$$b(\lambda_1 - \lambda_2)\mathbf{v_2} = \mathbf{0}$$

So

$$b(\lambda_1 - \lambda_2) = \mathbf{0}$$

But $\lambda_1 \neq \lambda_2$, so $\lambda_1 - \lambda_2 \neq 0$, hence we get $b = 0$.

But going back to the first equation, we get:

$$a\mathbf{v_1} = \mathbf{0}$$

So $a = 0$.

Hence $a = b = 0$, and we're done!

(d) If a matrix A has orthogonal columns, then it is an orthogonal matrix.

FALSE

Remember that an ortho**gonal** matrix has to have ortho**normal** columns!

(e) For every subspace W and every vector $\mathbf{y}$, $\mathbf{y} - Proj_W\mathbf{y}$ is orthogonal to $Proj_W\mathbf{y}$ (proof by picture is ok here)
TRUE

Draw a picture! $Proj_W\mathbf{y}$ is just another name for $\hat{y}$.

(f) If $\mathbf{y}$ is already in W, then $Proj_W\mathbf{y} = \mathbf{y}$

TRUE

Again, draw a picture!

If you want a more mathematical proof, here it is:

Let $\mathcal{B} = \{\mathbf{w_1}, \cdots \mathbf{w_p}\}$ be an orthogonal basis for W ($p = Dim(W)$).

Then $y = \left(\frac{\mathbf{y}\cdot\mathbf{w_1}}{\mathbf{w_1}\cdot\mathbf{w_1}}\right)\mathbf{w_1} + \cdots + \left(\frac{\mathbf{y}\cdot\mathbf{w_p}}{\mathbf{w_p}\cdot\mathbf{w_p}}\right)\mathbf{w_p}$.

But then, by definition of $Proj_W\mathbf{y} = \hat{\mathbf{y}}$, we get:

$$\hat{y} = \left(\frac{\mathbf{y} \cdot \mathbf{w_1}}{\mathbf{w_1} \cdot \mathbf{w_1}}\right) \mathbf{w_1} + \cdots + \left(\frac{\mathbf{y} \cdot \mathbf{w_p}}{\mathbf{w_p} \cdot \mathbf{w_p}}\right) \mathbf{w_p} = y$$

So $\hat{\mathbf{y}} = \mathbf{y}$ in this case.

2. *(20 points)* Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$, where:

$$A = \begin{bmatrix} 7 & -6 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 7 \end{bmatrix}$$

Eigenvalues: $det(A - \lambda I) = 0$ (expanding along last column), which gives $(\lambda - 1)(\lambda - 7)^2 = 0$, so $\boxed{\lambda = 1, 7}$

$\underline{\lambda = 1}$

$$Nul(A-I) = Nul\left(\begin{bmatrix} 6 & -6 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 6 \end{bmatrix}\right) = Nul\left(\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\right) = Span\left\{\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}\right\}$$

$\underline{\lambda = 7}$

$$Nul(A-7I) = Nul\left(\begin{bmatrix} 0 & -6 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}\right) = Nul\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = Span\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

Hence:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}, P = \begin{bmatrix} -2 & 1 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

3. *(10 points)* Define $T : P_3 \to P_3$ by:

$$T(p(t)) = tp''(t) - 2p'(t)$$

Find the matrix A of T relative to the basis $\mathcal{B} = \{1, t, t^2, t^3\}$ of P_3

First calculate:

- $T(1) = t(0) - 2(0) = 0$
- $T(t) = t(0) - 2(1) = -2$
- $T(t^2) = t(2) - 2(2t) = -2t$
- $T(t^3) = t(6t) - 2(3t^2) = 6t^2 - 6t^2 = 0$

Now evaluate all those vectors with respect to $\mathcal{B}$:

- $[T(1)]_{\mathcal{B}} = [0]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
- $[T(t)]_{\mathcal{B}} = [-2]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
- $[T(t^2)]_{\mathcal{B}} = [-2t]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}$
- $[T(t^3)]_{\mathcal{B}} = [0]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Putting everything together, we get that the matrix of T relative to $\mathcal{B}$ is:

$$A = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

4. *(15 points)* Let $\mathcal{B} = \left\{ \begin{bmatrix} 7 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}, \mathcal{C} = \left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$.

(a) Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$

We want to find $\mathcal{C} \overset{P}{\leftarrow} \mathcal{B}$.

$$\left[\ \mathcal{C} \mid \mathcal{B}\ \right] \to \left[\begin{array}{cc|cc} 4 & 5 & 7 & 2 \\ 1 & 2 & -2 & -1 \end{array}\right] \to \left[\begin{array}{cc|cc} 4 & 5 & 7 & 2 \\ 0 & -3 & 15 & 6 \end{array}\right] \to \left[\begin{array}{cc|cc} 4 & 5 & 7 & 2 \\ 0 & 1 & -5 & -2 \end{array}\right]$$
$$\to \left[\begin{array}{cc|cc} 4 & 0 & 32 & 12 \\ 0 & 1 & -5 & -2 \end{array}\right]$$
$$\to \left[\begin{array}{cc|cc} 1 & 0 & 8 & 3 \\ 0 & 1 & -5 & -2 \end{array}\right]$$

Hence:

$$\mathcal{C} \overset{P}{\leftarrow} \mathcal{B} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}$$

(b) Find $[\mathbf{x}]_{\mathcal{C}}$, where $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

We have:

$$[\mathbf{x}]_{\mathcal{C}} = \mathcal{C} \overset{P}{\leftarrow} \mathcal{B}\ [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 47 \\ -30 \end{bmatrix}$$

5. *(20 points, 10 points each)*

(a) Find a basis for $Row(A)$ and $Col(A)$, where:

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 0 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}$$

If you row-reduce A, you get that:

$$A \sim \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Note: You could have row-reduced it further, but no need!

Notice that the pivots are in the all 4 rows and the 1st, 3rd, 4th, and 5th column respectively, hence:

<u>Basis for Row(A):</u>

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -3 \\ 6 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

<u>Basis for Col(A):</u>

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b) What is $Rank(A)$? What is $Dim(Nul(A))$?

$Rank(A) = 4$ (number of pivots)

$Dim(Nul(A)) = 5 - Rank(A) = 5 - 4 = 1$ (by Rank-Nullity theorem)

6. *(15 points)*

(a) Find an invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that $A = PCP^{-1}$, where:

$$A = \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix}$$

<u>Eigenvalues:</u>

The characteristic polynomial of A is: $det(A - \lambda I) = (\lambda - 2)(\lambda) + 2 = \lambda^2 - 2\lambda + 2 = 0$ iff $\lambda = 1 \pm i$

<u>Eigenspace for $\lambda = 1 - i$</u>

$$Nul(A-(1-i)I) = Nul\left(\begin{bmatrix} 1+i & -2 \\ 1 & -1+i \end{bmatrix}\right) = Nul\left(\begin{bmatrix} 1 & -1+i \\ 0 & 0 \end{bmatrix}\right) = Span\left\{\begin{bmatrix} 1-i \\ 1 \end{bmatrix}\right\}$$

So an eigenvector corresponding to $\lambda = 1 - i$ is $\mathbf{v} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$

<u>Finding P and C:</u>

First of all, for P, we have:

$$Re(\mathbf{v}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, Im(\mathbf{v}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Hence: $P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$.

As for C, we have $Re(\lambda) = 1$, $Im(\lambda) = -1$. Now remember that you put those values on the first **ROW** of C, and you get:

$$C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

(Remember that the diagonal entries of C are equal and the nondiagonal ones are opposite of each other)

(b) Write C as a composition of a rotation and a scaling.

C is a rotation by ϕ followed by a scaling r.

r is given by: $r = \sqrt{det(C)} = \sqrt{2}$, hence:

$$C = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

If you compare this latter matrix with the rotation matrix $\begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$, you should realize that $\phi = \frac{\pi}{4}$.

Hence C is a rotation by $\boxed{\phi = \frac{\pi}{4}}$ followed by a scaling $\boxed{r = \sqrt{2}}$.