MATH 54 – MOCK MIDTERM 2 – SOLUTIONS

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1. (30 points, 5 pts each)

Label the following statements as T or F. Make sure to JUSTIFY YOUR ANSWERS!!! You may use any facts from the book or from lecture.

(a) If \( A = \{ a_1, a_2, a_3 \} \) and \( D = \{ d_1, d_2, d_3 \} \) are bases for \( V \), and \( P \) is the matrix whose \( i \)th column is \([d_i]_A\), then for all \( x \) in \( V \), we have \([x]_D = P [x]_A\)

FALSE

First of all, \( P = \begin{bmatrix} [d_1]_A & [d_2]_A & [d_3]_A \end{bmatrix} = A^P \leftarrow D \) (remember, you always evaluate with respect to the new, cool basis, here it is \( A \)), so we should have:

\[ [x]_A = A^P \leftarrow D \ [x]_D = P [x]_D \]

And not the opposite!

(b) A 3 \( \times \) 3 matrix \( A \) with only one eigenvalue cannot be diagonalizable

SUPER FALSE!!!!!!!!!

Remember that to check if a matrix is not diagonalizable, you really have to look at the eigenvectors!

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For example, \( A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \) has only eigenvalue 2, but is diagonalizable (it’s diagonal!). Or you can choose \( A \) to be the \( O \) matrix, or the identity matrix, this also works!

(c) If \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are 2 eigenvectors of \( A \) corresponding to 2 different eigenvalues \( \lambda_1 \) and \( \lambda_2 \), then \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are linearly independent!

\textbf{TRUE} (finally!)

\textbf{Note:} The proof is a bit complicated, but I’ve seen this on a past exam! I think at that point, the professor wanted to get revenge on his students for not coming to lecture!

Remember that eigenvectors have to be nonzero!

Now, assume \( a\mathbf{v}_1 + b\mathbf{v}_2 = 0 \).

Then apply \( A \) to this to get:

\[ A(a\mathbf{v}_1 + b\mathbf{v}_2) = A(0) = 0 \]

That is:

\[ aA(\mathbf{v}_1) + bA(\mathbf{v}_2) = 0 \]

\[ a\lambda_1\mathbf{v}_1 + b\lambda_2\mathbf{v}_2 = 0 \]

However, we can also multiply the original equation by \( \lambda_1 \) to get:

\[ a\lambda_1\mathbf{v}_1 + b\lambda_1\mathbf{v}_2 = 0 \]

Subtracting this equation from the one preceding it, we get:

\[ b(\lambda_1 - \lambda_2)\mathbf{v}_2 = 0 \]

So

\[ b(\lambda_1 - \lambda_2) = 0 \]
But $\lambda_1 \neq \lambda_2$, so $\lambda_1 - \lambda_2 \neq 0$, hence we get $b = 0$.

But going back to the first equation, we get:

$$a v_1 = 0$$

So $a = 0$.

Hence $a = b = 0$, and we’re done!

(d) If a matrix $A$ has orthogonal columns, then it is an orthogonal matrix.

**FALSE**

Remember that an orthogonal matrix has to have orthonormal columns!

(e) For every subspace $W$ and every vector $y$, $y - \text{Proj}_W y$ is orthogonal to $\text{Proj}_W y$ (proof by picture is ok here)

**TRUE**

Draw a picture! $\text{Proj}_W y$ is just another name for $\hat{y}$.

(f) If $y$ is already in $W$, then $\text{Proj}_W y = y$

**TRUE**

Again, draw a picture!

If you want a more mathematical proof, here it is:

Let $B = \{w_1, \cdots, w_p\}$ be an orthogonal basis for $W$ ($p = \text{Dim}(W)$).

Then $y = \left(\frac{y \cdot w_1}{w_1 \cdot w_1}\right) w_1 + \cdots + \left(\frac{y \cdot w_p}{w_p \cdot w_p}\right) w_p$.

But then, by definition of $\text{Proj}_W y = \hat{y}$, we get:
$$\hat{y} = \left( \frac{y \cdot w_1}{w_1 \cdot w_1} \right) w_1 + \cdots + \left( \frac{y \cdot w_p}{w_p \cdot w_p} \right) w_p = y$$

So $\hat{y} = y$ in this case.
2. (20 points) Find a diagonal matrix $D$ and an invertible matrix $P$ such that $A = PDP^{-1}$, where:

$$A = \begin{bmatrix} 7 & -6 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 7 \end{bmatrix}$$

**Eigenvalues:** $det(A - \lambda I) = 0$ (expanding along last column), which gives $(\lambda - 1)(\lambda - 7)^2 = 0$, so $\lambda = 1, 7$

$\lambda = 1$

$Nul(A - I) = Nul \begin{bmatrix} 6 & -6 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 6 \end{bmatrix} = Nul \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = Span \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$

$\lambda = 7$

$Nul(A - 7I) = Nul \begin{bmatrix} 0 & -6 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} = Nul \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = Span \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Hence:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}, P = \begin{bmatrix} -2 & 1 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$
3. (10 points) Define $T : P_3 \rightarrow P_3$ by:

$$T(p(t)) = tp''(t) - 2p'(t)$$

Find the matrix $A$ of $T$ relative to the basis $B = \{1, t, t^2, t^3\}$ of $P_3$

First calculate:

- $T(1) = t(0) - 2(0) = 0$
- $T(t) = t(0) - 2(1) = -2$
- $T(t^2) = t(2) - 2(2t) = -2t$
- $T(t^3) = t(6t) - 2(3t^2) = 6t^2 - 6t^2 = 0$

Now evaluate all those vectors with respect to $B$:

- $[T(1)]_B = [0]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
- $[T(t)]_B = [-2]_B = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
- $[T(t^2)]_B = [-2t]_B = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}$
- $[T(t^3)]_B = [0]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Putting everything together, we get that the matrix of $T$ relative to $B$ is:
4. (15 points) Let $B = \left\{ \begin{bmatrix} 7 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}, C = \left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$.

(a) Find the change-of-coordinates matrix from $B$ to $C$

We want to find $P_{C \leftarrow B}$.

\[
\begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix} 
\begin{bmatrix} 7 \\ -2 \end{bmatrix} 
\begin{bmatrix} 2 \\ -1 \end{bmatrix} 
\begin{bmatrix} 4 & 5 \\ 0 & 1 \end{bmatrix} 
\begin{bmatrix} 7 & 2 \\ 15 & 6 \end{bmatrix} 
\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} 
\begin{bmatrix} 32 & 12 \\ -5 & -2 \end{bmatrix} 
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} 
\begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}
\]

Hence:

\[
P_{C \leftarrow B} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}
\]

(b) Find $[x]_C$, where $[x]_B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

We have:

\[
[x]_C = P_{C \leftarrow B} [x]_B = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 47 \\ -30 \end{bmatrix}
\]
5. (20 points, 10 points each)

(a) Find a basis for $\text{Row}(A)$ and $\text{Col}(A)$, where:

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 0 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}$$

If you row-reduce $A$, you get that:

$$A \sim \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

**Note:** You could have row-reduced it further, but no need!

Notice that the pivots are in the all 4 rows and the 1st, 3rd, 4th, and 5th column respectively, hence:

**Basis for Row(A):**

$$B = \begin{bmatrix} 2 \\ -3 \\ 6 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

**Basis for Col(A):**

$$B = \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$
(b) What is $\text{Rank}(A)$? What is $\text{Dim}(\text{Nul}(A))$?

$\text{Rank}(A) = 4$ (number of pivots)

$\text{Dim}(\text{Nul}(A)) = 5 - \text{Rank}(A) = 5 - 4 = 1$ (by Rank-Nullity theorem)

6. (15 points)

(a) Find an invertible matrix $P$ and a matrix $C$ of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that $A = PCP^{-1}$, where:

$$A = \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix}$$

Eigenvalues:

The characteristic polynomial of $A$ is: $\det(A - \lambda I) = (\lambda - 2)(\lambda) + 2 = \lambda^2 - 2\lambda + 2 = 0$ iff $\lambda = 1 \pm i$

Eigenvalue for $\lambda = 1 - i$

$$\text{Nul}(A-(1-i)I) = \text{Nul}\left(\begin{bmatrix} 1+i & -2 \\ 1 & -1+i \end{bmatrix}\right) = \text{Nul}\left(\begin{bmatrix} 1 & -1+i \\ 0 & 0 \end{bmatrix}\right) = \text{Span}\left\{\begin{bmatrix} 1-i \\ 1 \end{bmatrix}\right\}$$

So an eigenvector corresponding to $\lambda = 1 - i$ is $v = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$

Finding $P$ and $C$:

First of all, for $P$, we have:

$$\text{Re}(v) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{Im}(v) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
Hence:  \( P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \).

As for \( C \), we have \( Re(\lambda) = 1, Im(\lambda) = -1 \). Now remember that you put those values on the first \textbf{ROW} of \( C \), and you get:

\[
C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}
\]

(Remember that the diagonal entries of \( C \) are equal and the nondiagonal ones are opposite of each other)

(b) Write \( C \) as a composition of a rotation and a scaling.

\( C \) is a rotation by \( \phi \) followed by a scaling \( r \).

\( r \) is given by: \( r = \sqrt{\det(C)} = \sqrt{2} \), hence:

\[
C = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}
\]

If you compare this latter matrix with the rotation matrix \( \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \), you should realize that \( \phi = \frac{\pi}{4} \).

Hence \( C \) is a rotation by \( \phi = \frac{\pi}{4} \) followed by a scaling \( r = \sqrt{2} \).