MATH 54 – MOCK MIDTERM 1 – SOLUTIONS

PEYAM RYAN TABRIZIAN

1. *(45 points, 5 points each)*

(a) If A and B are square matrices, then $(A+B)^{-1} = A^{-1} + B^{-1}$.

FALSE

For example, take $A = [2]$ and $B = [3]$. Then the statement says: Is $\frac{1}{2+3} = \frac{1}{2} + \frac{1}{3}$? Which is not true.

Other explanation: Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, then $A + B$ is the zero matrix, whose inverse is not defined, while the right-hand-side gives you 0.

(b) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a one-to-one linear transformation, then T is also onto.

TRUE

Let A be the matrix of T. Then, if T is one-to-one, then A is invertible (by one of the conditions of invertibility), and hence, by another condition of invertibility, this implies that T is onto. Note that is works precisely because $m = n$, the result doesn't hold in general!

Date: Thursday, September 25th, 2014.

(c) If $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ are linearly independent vectors in $\mathbb{R}^n$, then $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent as well!

TRUE

Suppose $a\mathbf{v_1} + b\mathbf{v_2} = \mathbf{0}$.

Goal: We want to show $a = b = 0$.

Now here's a clever trick: Add $0\mathbf{v_3} = \mathbf{0}$ to both sides of the equation.

Then we get: $a\mathbf{v_1} + b\mathbf{v_2} + 0\mathbf{v_3} = \mathbf{0}$

In particular, if we let $c = 0$, then we get: $a\mathbf{v_1} + b\mathbf{v_2} + c\mathbf{v_3} = \mathbf{0}$

But $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly independent, so $a = b = c = 0$.

In particular $a = b = 0$, which we wanted to show!

Note: I have to admit, this is a tricky proof! But it illustrates why it's important to write down what you want to show and what you know!

(d) If A is an invertible square matrix, then $(A^T)^{-1} = (A^{-1})^T$

TRUE

Let $B = (A^{-1})^T$. All we need to show is that $A^T B = BA^T = I$, because then $B = \left(A^T\right)^{-1}$, which is what we want to show.

But:

$$A^T B = A^T \left(A^{-1}\right)^T = \left(A^{-1}A\right)^T = I^T = I$$

Where in the first step, we used the property of transposes $(CD)^T = D^T C^T$.

Similarly:

$$BA^T = \left(A^{-1}\right)^T A^T = \left(AA^{-1}\right)^T = I^T = I$$

Hence $A^T B = BA^T = I$, which is what we needed to show!

Note: This question is hard too! The reason I put this question is because Prof. Grunbaum mentioned in lecture that it might be on the midterm!

(e) If A is a 3×3 matrix with two pivot positions, then the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

TRUE

If A has two pivot positions, then it has a row of zeros, and hence, because A is a 3×3 matrix, the solution $A\mathbf{x} = \mathbf{0}$ has at least one free variable, hence the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution!

(f) If A and B are square matrices, then $det(A + B) = det(A) + det(B)$.

FALSE

For example, take $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Then $det(A) = 1$, $det(B) = 1$, but $det(A + B) = det(O) = 0$ (where O is the zero-matrix).

(g) If $Nul(A) = \{\mathbf{0}\}$, then A is invertible.

FALSE

Don't worry, this got me too! This statement *is* true if A is **SQUARE** ! But if A is not square, this statement is never true!

For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $Nul(A) = \{\mathbf{0}\}$, but A is not invertible, because it is not square.

(h) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$

FALSE!

$\mathbb{R}^2$ is not even a *subset* of $\mathbb{R}^3$!!! Don't confuse this with $\left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$, which *is* a subspace of $\mathbb{R}^3$ and very similar to $\mathbb{R}^2$ (but not exactly the same)

(i) If W is a subspace of V and $\mathcal{B}$ is a basis for V, then some subset of $\mathcal{B}$ is a basis for W.

FALSE

This is also very tricky (this got me too :)), because the 'opposite' statement does hold, namely if $\mathcal{B}$ is a basis for W, you can always complete $\mathcal{B}$ to become a basis of V (this is the 'basis extension theorem').

As a counterexample, take $V = \mathbb{R}^3$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, and $W = Span \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ (a line in $\mathbb{R}^3$).

If the statement was true, then one of the vectors in $\mathcal{B}$ would be a basis for W, but this is bogus.

2. *(15 points)* Solve the following system (or say it has no solutions):

$$\begin{cases} x+y+z=0 \\ 2x+2z=0 \\ 3x+y+3z=0 \end{cases}$$

Write down the augmented matrix and row-reduce:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 3 & 1 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now rewrite this as a system:

$$\begin{cases} x+z=0 \\ y=0 \end{cases}$$

That is:

$$\begin{cases} x=-z \\ y=0 \\ z=z \end{cases}$$

Or in vector form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

(where z is free)

3. *(20 points)* Find the inverse of the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Form the (super) augmented matrix and row-reduce:

$$\begin{aligned}
\begin{bmatrix} A & I \end{bmatrix} &= \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 2 & 1 & -1 & 1 & 0 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 3 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{bmatrix} \\
&= \begin{bmatrix} I & A^{-1} \end{bmatrix}
\end{aligned}$$

Hence:

$$A^{-1} = \begin{bmatrix} -1 & -1 & 3 \\ -1 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

4. *(10 points)* What's the next elementary row operation you would use to transform the following matrix in row-echelon form? What is the corresponding elementary matrix?

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

Notice that A is almost in row-echelon form, except for that 2 in the last row. Hence, we want to subtract 2 times the second row from the third row to 'eliminate' the 2.

In other words, the answer is: $\boxed{\text{Add } (-2) \text{ times the } 2^{nd} \text{ row to the } 3^{rd} \text{ row}}$.

The corresponding elementary matrix is:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Optional: You can indeed check that:

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

which is indeed in row-echelon form!

5. *(10 points, 5 points each)* Evaluate the following products if they are defined, or say 'undefined'

(a) AB, where:

$$A = \begin{bmatrix} 2 & 5 \\ 0 & 7 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

(b) AB, where:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 3 \\ -2 & 1 & -3 \end{bmatrix}$$

6. *(10 points)* Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation which reflects points in the plane about the origin.

(a) (5 points) Find the matrix A of T.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\0\end{bmatrix}, T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\-1\end{bmatrix}$$

Hence:

$$A = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$$

(b) (5 points) Use A to find $T(1,1)$.

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix}$$

7. *(10 points)* Find the determinant of the following matrix A:

$$A = \begin{bmatrix} 1 & 42 & 536 & 789 & 4201 & 123456789 \\ 0 & 1 & 2011 & 2012 & \pi m & \text{Dolphin} \\ 0 & 0 & 2 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 2 & -1 \end{bmatrix}$$

Note: The answer may surprise you :)

First of all, expanding along the first column, and then along the first column again, we get that:

$$det(A) = \begin{vmatrix} 2 & 0 & 4 & 5 \\ 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 2 & -1 \end{vmatrix}$$

Now expanding along the 3rd row (or the second column), we get:

$$det(A) = -\begin{vmatrix} 2 & 4 & 5 \\ 0 & 3 & 1 \\ 4 & 2 & -1 \end{vmatrix}$$

Note: Careful about the signs!

Finally, expanding along the second row (or first column), we get:

$$det(A) = -\left(3\begin{vmatrix} 2 & 5 \\ 4 & -1 \end{vmatrix} - \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix}\right) = -\left((3)(-22) + 12\right) = 54$$

NO WAY!!! I know, right? I did not expect that at all! :D

Note: Here's a smarter way to evaluate $det(A)$ (courtesy Rongchang Lei): Just row-reduce!

$$
\begin{aligned}
det(A) &= \begin{vmatrix} 2 & 0 & 4 & 5 \\ 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 2 & -1 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 0 & 4 & 5 \\ 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & -11 \end{vmatrix} \qquad (R_4 - 2R_1) \\
&= - \begin{vmatrix} 2 & 0 & 4 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -6 & -11 \end{vmatrix} \qquad (R_2 \leftrightarrow R_3) \\
&= - \begin{vmatrix} 2 & 0 & 4 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -9 \end{vmatrix} \qquad (R_4 - 2R_3) \\
&= -(2)(1)(3)(-9) \qquad \text{upper triangular matrix} \\
&= 54
\end{aligned}
$$

8. *(20 points, 10 points each)*

(a) Find a basis for $Col(A)$, where:

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 0 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}$$

If you row-reduce A, you get that:

$$A \sim \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Note: You could have row-reduced it further, but no need!

Notice that the pivots are in the all 4 rows and the 1st, 3rd, 4th, and 5th column respectively, hence:

Basis for Col(A):

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b) What is $Rank(A)$? What is $Dim(Nul(A))$?

$Rank(A) = 4$ (number of pivots)

$Dim(Nul(A)) = 5 - Rank(A) = 5 - 4 = 1$ (by Rank-Nullity theorem)

9. *(10 points)* Find a basis for $Nul(A)$ and $Col(A)$, where A is the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Row-reduce A:

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Notice that every column has a pivot, hence to get a basis for $Col(A)$, go back to A and select all the columns for A, and you get that a basis for $Col(A)$ is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\}$$

Finally, to find a basis for $Nul(A)$, we need to solve $A\mathbf{x} = \mathbf{0}$. However, notice that A is a 3×3 matrix with 3 pivots, hence the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. It follows that:

$$Nul(A) = \{\mathbf{0}\}$$

Note: You might be tempted to say that $\{\mathbf{0}\}$ is a basis for $Nul(A)$, but this is technically wrong (but you wouldn't get points off for that). The correct answer is that the basis is $= \emptyset$, but you don't need to know that. The main point is that you should be able to know the procedure for finding $Nul(A)$.