MATH 54 – MOCK FINAL EXAM – SOLUTIONS

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1. *(20 points)* Use the Gram-Schmidt process to obtain an orthonormal basis of $W = Span\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$, where:

$$\mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 6 \\ -3 \\ 1 \\ 11 \end{bmatrix}$$

First of all, $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ is an orthogonal basis for W, where:

$$\mathbf{w_1} = \mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\mathbf{w_2} = \mathbf{v_2} - \left(\frac{\mathbf{v_2} \cdot \mathbf{w_1}}{\mathbf{w_1} \cdot \mathbf{w_1}}\right)\mathbf{w_1} = \mathbf{v_2} - \frac{7}{14}\mathbf{w_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 0 \\ 3 \\ 0 \\ -1 \end{bmatrix}$$

Date: Friday, December 9th, 2011.

$$
\begin{aligned}
\mathbf{w_3} = \mathbf{v_3} - \left(\frac{\mathbf{v_3}\cdot\mathbf{w_1}}{\mathbf{w_1}\cdot\mathbf{w_1}}\right)\mathbf{w_1} - \left(\frac{\mathbf{v_3}\cdot\mathbf{w_2}}{\mathbf{w_2}\cdot\mathbf{w_2}}\right)\mathbf{w_2} =& \mathbf{v_3} - \frac{42}{14}\mathbf{w_1} - \frac{-20}{10}\mathbf{w_2} \\
=& \mathbf{v_3} - 3\mathbf{w_1} + 2\mathbf{w_2} = \begin{bmatrix} 6 \\ -3 \\ 1 \\ 11 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 3 \\ 0 \\ -1 \end{bmatrix} \\
=& \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}
\end{aligned}
$$

Finally, an orthonormal basis for W is $\{\mathbf{w_1}', \mathbf{w_2}', \mathbf{w_3}'\}$, where:

$$
\mathbf{w_1}' = \frac{\mathbf{w_1}}{\|\mathbf{w_1}\|} = \frac{1}{\sqrt{14}}\begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \mathbf{w_2}' = \frac{\mathbf{w_2}}{\|\mathbf{w_2}\|} = \frac{1}{\sqrt{10}}\begin{bmatrix} 0 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \mathbf{w_3}' = \frac{\mathbf{w_3}}{\|\mathbf{w_3}\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}
$$

2. *(10 points)* Find a least squares solution to the following system $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A^T A\hat{\mathbf{x}} = A^T \mathbf{b}$$

$$\begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Which gives:

$$\hat{\mathbf{x}} = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{5}{9} \\ \frac{1}{9} \end{bmatrix}$$

3. *(10 points)* Find the orthogonal projection of t^2 onto the subspace W spanned by $\{1, t\}$, with respect to the following inner product:

$$\langle p, q \rangle = \int_{-1}^{1} p(t)q(t)dt$$

Let $p_1(t) = 1$, $p_2(t) = t$, $p_3(t) = t^2$, then:

$$\hat{p_3} = \left(\frac{\langle p_3, p_1 \rangle}{\langle p_1, p_1 \rangle}\right) p_1 + \left(\frac{\langle p_3, p_2 \rangle}{\langle p_2, p_2 \rangle}\right) p_2 = \left(\frac{\int_{-1}^{1} t^2 dt}{\int_{-1}^{1} dt}\right)(1) + \left(\frac{\int_{-1}^{1} t^3 dt}{\int_{-1}^{1} t^2 dt}\right)(t) = \left(\frac{\frac{2}{3}}{2}\right)(1) + \left(\frac{0}{\frac{2}{3}}\right) t = \frac{1}{3}$$

$$\hat{p_3}(t) = \frac{1}{3}$$

4. *(20 points)*

Find a diagonal matrix D and an orthogonal matrix P such that $A = PDP^T$, where:

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

Eigenvalues:

$$det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -2 & 4 \\ -2 & 6-\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda) \begin{vmatrix} 6-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} + 2 \begin{vmatrix} -2 & 2 \\ 4 & 3-\lambda \end{vmatrix} + 4 \begin{vmatrix} -2 & 6-\lambda \\ 4 & 2 \end{vmatrix}$$

$$= (3-\lambda)\left((6-\lambda)(3-\lambda) - 4\right) + 2\left((-2)(3-\lambda) - 8\right) + 4\left(-4 - 4(6-\lambda)\right)$$

$$= -\lambda^3 + 12\lambda^2 - 21\lambda - 98$$

Now, using the rational roots theorem (the possible zeros are $\pm 1, \pm 2, \pm 7, \pm 14, \pm 49, \pm 98$), we get that $\lambda = -2$ is an eigenvalue, and using long division, we get:

$$-\lambda^3 + 12\lambda^2 - 21\lambda - 98 = -(\lambda - 2)(\lambda^2 - 14\lambda + 49) = -(\lambda - 2)(\lambda - 7)^2 = 0$$

Hence the eigenvalues are $\boxed{\lambda = -2, 7}$

Eigenvectors

$\lambda = -2$:

$$Nul(A+2I) = Nul\begin{bmatrix} 5 & 2 & -4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} = Nul\begin{bmatrix} 5 & -2 & 4 \\ 1 & -4 & -1 \\ 0 & 18 & 9 \end{bmatrix}$$
$$= Nul\begin{bmatrix} 1 & -4 & -1 \\ 0 & 18 & 9 \\ 0 & 18 & 9 \end{bmatrix} = Nul\begin{bmatrix} 1 & -4 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
$$= Nul\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$
$$= Span\left\{\begin{bmatrix} -1 \\ \frac{-1}{2} \\ 1 \end{bmatrix}\right\} = Span\left\{\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}\right\}$$

$\underline{\lambda = 7}$:

$$Nul(A-7I) = Nul\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix} = Nul\begin{bmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$$
$$= Nul\begin{bmatrix} 2 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$= Span\left\{\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\} = Span\left\{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

<u>Gram-Schmidt</u>

$\underline{\lambda = -2}$:

Let $\mathbf{u_1} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, then $\mathbf{v_1} = \mathbf{u_1}$, and:

$$\mathbf{w_1} = \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{-2}{3} \end{bmatrix}$$

$\underline{\lambda = 7}$:

Let $\mathbf{u_1} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Then:

$$\mathbf{v_1} = \mathbf{u_1} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$\mathbf{v_2} = \mathbf{u_2} - \frac{\mathbf{u_2} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{5} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{2}{5} \\ 1 \end{bmatrix} \sim \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

And finally:

$$\mathbf{w_1} = \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \\ 0 \end{bmatrix}, \mathbf{w_2} = \frac{\mathbf{v_2}}{\|\mathbf{v_2}\|} = \begin{bmatrix} \frac{4}{\sqrt{45}} \\ \frac{2}{\sqrt{45}} \\ \frac{5}{\sqrt{45}} \end{bmatrix}$$

<u>Conclusion:</u>
Putting everything together, we get $A = PDP^T$, where:

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}, P = \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{45}} \\ \frac{1}{3} & \frac{-2}{\sqrt{5}} & \frac{2}{\sqrt{45}} \\ \frac{-2}{3} & 0 & \frac{5}{\sqrt{45}} \end{bmatrix}$$

5. *(20 points, 2 points each)*

Mark the following statements as **TRUE** or **FALSE**. If the statement is **TRUE**, don't do anything. If the statement is **FALSE**, provide an explicit counterexample.

(a) If A is a 3×3 matrix with eigenvalues $\lambda = 0, 2, 3$, then A must be diagonalizable!

TRUE (an $n \times n$ matrix with 3 distinct eigenvalues is diagonalizable)

(b) There does not exist a 3×3 matrix A with eigenvalues $\lambda = 1, -1, -1 + i$.

TRUE (here we assume A has real entries; eigenvalues always come in complex conjugate pairs, i.e. if A has eigenvalue $-1+i$, it must also have eigenvalue $-1 - i$)

(c) If A is a symmetric matrix, then all its eigenvectors are orthogonal.

FALSE: Take A to be your favorite symmetric matrix, and, for example, take $\mathbf{v}$ to be one eigenvector, and $\mathbf{w}$ to be the *same* eigenvector (or a different eigenvector corresponding to the same eigenvalue). That's why we had to apply the Gram Schmidt process to each eigenspace in the previous problem!

(d) If Q is an orthogonal $n \times n$ matrix, then $Row(Q) = Col(Q)$.

TRUE: (since Q is orthogonal, $Q^T Q = I$, so Q is invertible, hence $Row(Q) = Col(Q) = \mathbb{R}^n$)

(e) The equation $A\mathbf{x} = \mathbf{b}$, where A is a $n \times n$ matrix always has a unique least-squares solution.

FALSE: Take A to be the zero matrix, and $\mathbf{b}$ to be the zero vector! This statement is true if A has rank n.

(f) If $AB = I$, then $BA = I$.

FALSE: Let $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $AB = I$, but $BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$!

(g) If A is a square matrix, then $Rank(A) = Rank(A^2)$

FALSE: Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $Rank(A) = 1$, but $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $Rank(A^2) = 0$.

(h) If W is a subspace, and $P\mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto W, then $P^2\mathbf{y} = P\mathbf{y}$

TRUE (draw a picture! If you orthogonally project $P\mathbf{y} = \hat{\mathbf{y}}$ on W, you get $\hat{\mathbf{y}}$)

(i) If $T : V \to W$, where $dim(V) = 3$ and $dim(W) = 2$, then T cannot be one-to-one.

TRUE (by Rank-Nullity theorem, $dim(Nul(T)) + Rank(T) = 3$. But $Rank(T)$ can only be at most $dim(W) = 2$, so $dim(Nul(T)) > 0$, so $Nul(T) \neq \{\mathbf{0}\}$)

(j) If A is similar to B, then $det(A) = det(B)$.

TRUE (If $A = PBP^{-1}$, then $det(A) = det(B)$)

6. *(20 points)* Solve the following system $\mathbf{x}' = A\mathbf{x}$, where:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

<u>Eigenvalues:</u>

$$det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda)\left((1-\lambda)^2 + 1\right) = 0$$

which gives you $\lambda = 1, 1 \pm i$.

<u>Eigenvectors:</u>

$\underline{\lambda = 1}$:

$$Nul(A - I) = Nul \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = Nul \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = Span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$\underline{\lambda = 1 + i}$

$$\begin{aligned} Nul(A - (1+i)I) &= Nul \begin{bmatrix} -i & 2 & -1 \\ 0 & -i & 1 \\ 0 & -1 & -i \end{bmatrix} \\ &= Nul \begin{bmatrix} 1 & 2i & -i \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix} \\ &= Nul \begin{bmatrix} 1 & 0 & 2-i \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix} \\ &= Span \left\{ \begin{bmatrix} -2+i \\ -i \\ 1 \end{bmatrix} \right\} \end{aligned}$$

Note: Here we used the fact that $\frac{1}{i} = \frac{i}{i^2} = -i$.

Now separate the eigenvector into real and imaginary parts:

$$\begin{bmatrix} -2+i \\ -i \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

<u>Solution:</u> Hence our solution is:

$$\mathbf{x}(t) = Ae^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + B \left(e^t \cos(t) \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - e^t \sin(t) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right) + C \left(e^t \sin(t) \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + e^t \cos(t) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right)$$

7. *(10 points)* Solve the following system $\mathbf{x}' = A\mathbf{x}$, where:

$$A = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$$

<u>Eigenvalues:</u>

$$det(A-\lambda I) = \begin{bmatrix} 1-\lambda & -1 \\ 4 & -3-\lambda \end{bmatrix} = (1-\lambda)(-3-\lambda)+4 = \lambda^2+2\lambda+1 = (\lambda+1)^2 = 0$$

which gives $\lambda = -1$.

<u>Eigenvectors</u>

$$Nul(A+I) = Nul \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} = Nul \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} = Span \left\{ \begin{bmatrix} 1 & 2 \end{bmatrix} \right\}$$

<u>Generalized eigenvector:</u>

Now find $\mathbf{u}$ such that $(A+I)\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$:

$$\begin{bmatrix} 2 & -1 & | \; 1 \\ 4 & -2 & | \; 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & | \; 1 \\ 0 & 0 & | \; 0 \end{bmatrix} = \left\{ \begin{bmatrix} 0 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mid s \in \mathbb{R} \right\}$$

Now let $s = 0$, and you get $\mathbf{u} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

<u>Solution:</u>

$$\mathbf{x}(t) = Ae^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + B \left(te^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{-t} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

8. *(20 points)* Solve the following heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & 0 < x < 1, \quad t > 0 \\ u(0,t) = u(1,t) = 0 & t > 0 \\ u(x,0) = x & 0 < x < 1 \end{cases}$$

Step 1: Separation of variables. Suppose:

(1) $$u(x,t) = X(x)T(t)$$

Plug (1) into the differential equation (), and you get:

$$\begin{aligned} (X(x)T(t))_t =& (X(x)T(t))_{tt} \\ X(x)T'(t) =& X''(x)T(t) \end{aligned}$$

Rearrange and get:

(2) $$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}$$

Now $\frac{X''(x)}{X(x)}$ *only* depends on x, but by (2) *only* depends on t, hence it is constant:

(3) $$\begin{aligned} \frac{X''(x)}{X(x)} =& \lambda \\ X''(x) =& \lambda X(x) \end{aligned}$$

Also, we get:

(4) $$\begin{aligned} \frac{T'(t)}{T(t)} =& \lambda \\ T'(t) =& \lambda T(t) \end{aligned}$$

but we'll only deal with that later (Step 4)

Step 2: Consider (3):

$$X''(x) = \lambda X(x)$$

Note: Always start with $X(x)$, do **NOT** touch $T(t)$ until right at the end!

Now use the **boundary conditions** in ():

$$u(0,t) = X(0)T(t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0$$

$$u(1,t) = X(1)T(t) = 0 \Rightarrow X(1)T(t) = 0 \Rightarrow X(1) = 0$$

Hence we get:

$$(5) \qquad \begin{cases} X''(x) = \lambda X(x) \\ X(0) = 0 \\ X(1) = 0 \end{cases}$$

Step 3: Eigenvalues/Eigenfunctions. The auxiliary polynomial of (5) is $p(\lambda) = r^2 - \lambda$

Now we need to consider 3 cases:

<u>Case 1:</u> $\lambda > 0$, then $\lambda = \omega^2$, where $\omega > 0$

Then:

$$r^2 - \lambda = 0 \Rightarrow r^2 - \omega^2 = 0 \Rightarrow r = \pm\omega$$

Therefore:

$$X(x) = Ae^{\omega x} + Be^{-\omega x}$$

Now use $X(0) = 0$ and $X(1) = 0$:

$$X(0) = 0 \Rightarrow A + B = 0 \Rightarrow B = -A \Rightarrow X(x) = Ae^{\omega x} - Ae^{-\omega x}$$

$$X(1) = 0 \Rightarrow Ae^{\omega} - Ae^{-\omega} = 0 \Rightarrow Ae^{\omega} = Ae^{-\omega} \Rightarrow e^{\omega} = e^{-\omega} \Rightarrow \omega = -\omega \Rightarrow \omega = 0$$

But this is a **contradiction**, as we want $\omega > 0$.

Case 2: $\lambda = 0$, then $r = 0$, and:

$$X(x) = Ae^{0x} + Bxe^{0x} = A + Bx$$

And:

$$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = Bx$$

$$X(1) = 0 \Rightarrow B = 0 \Rightarrow X(x) = 0$$

Again, a **contradiction** (we want $X \not\equiv 0$, because otherwise $u(x,t) \equiv 0$)

Case 3: $\lambda < 0$, then $\lambda = -\omega^2$, and:

$$r^2 - \lambda = 0 \Rightarrow r^2 + \omega^2 = 0 \Rightarrow r = \pm \omega i$$

Which gives:

$$X(x) = A\cos(\omega x) + B\sin(\omega x)$$

Again, using $X(0) = 0$, $X(1) = 0$, we get:

$$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = B\sin(\omega x)$$

$$X(1) = 0 \Rightarrow B\sin(\omega) = 0 \Rightarrow \sin(\omega) = 0 \Rightarrow \omega = \pi m, \quad (m = 1, 2, \cdots)$$

This tells us that:

$$\begin{aligned} &\text{Eigenvalues:} \lambda = -\omega^2 = -(\pi m)^2 \quad (m = 1, 2, \cdots) \\ &\text{Eigenfunctions:} X(x) = \sin(\omega x) = \sin(\pi m x) \end{aligned} \tag{6}$$

Step 4: Deal with (4), and remember that $\lambda = -(\pi m)^2$:

$$T'(t) = \lambda T(t) \Rightarrow T(t) = Ae^{\lambda t} = T(t) = \widetilde{A_m} e^{-(\pi m)^2 t} \qquad m = 1, 2, \cdots$$

Note: Here we use $\widetilde{A_m}$ to emphasize that $\widetilde{A_m}$ depends on m.

Step 5: Take linear combinations:

$$u(x,t) = \sum_{m=1}^{\infty} T(t)X(x) = \sum_{m=1}^{\infty} \widetilde{A_m} e^{-(\pi m)^2 t} \sin(\pi m x) \tag{7}$$

Step 6: Use the initial condition $u(x,0) = x$ in ():

$$u(x,0) = \sum_{m=1}^{\infty} \widetilde{A_m} \sin(\pi m x) = x \qquad \text{on}(0,1) \tag{8}$$

Now we want to express x as a linear combination of sines, so we have to use a **sine series** (that's why we used $\widetilde{A_m}$ instead of A_m):

$$\begin{aligned}
\widetilde{A_m} &= \frac{2}{1}\int_0^1 x \sin(\pi m x) dx \\
&= 2\left(\left[-x\frac{\cos(\pi m x)}{\pi m}\right]_0^1 - \int_0^1 -\frac{\cos(\pi m x)}{\pi m} dx\right) \\
&= 2\left(-\frac{\cos(\pi m)}{\pi m} + \int_0^1 \frac{\cos(\pi m x)}{\pi m} dx\right) \\
&= 2\left(-\frac{(-1)^m}{\pi m} + \left[\frac{\sin(\pi m x)}{(\pi m)^2}\right]_0^1\right) \\
&= \frac{2(-1)^{m+1}}{\pi m} \qquad (m = 1, 2, \cdots)
\end{aligned}$$

Step 7: Conclude using (9)

$$u(x,t) = \sum_{m=1}^{\infty} \frac{2(-1)^{m+1}}{\pi m} e^{-(\pi m)^2 t} \sin(\pi m x) \tag{9}$$