MATH 54 – MIDTERM 2

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Name: ____

Note: The reason this exam is so long is because it combines half of the second midterm, half of the third midterm, and half of the final exam that I gave during Summer 2012.

1		16
2		15
3		10
4		10
5		10
6		30
7		4
8		5
Total		100

Date: Friday, July 27th, 2012.

1. *(16 points, 2 points each)*

Label the following statements as **T** or **F**. **Write your answers in the box below!**

NOTE: In this question, you do **NOT** have to show your work! Don't spend *too* much time on each question!

(a) If A is diagonalizable, then A^3 is diagonalizable.

(b) If A is a 3×3 matrix with 3 (linearly independent) eigenvectors, then A is diagonalizable

(c) If A is a 3×3 matrix with eigenvalues $\lambda = 1, 2, 3$, then A is invertible

(d) If A is a 3×3 matrix with eigenvalues $\lambda = 1, 2, 3$, then A is (always) diagonalizable

(e) If A is a 3×3 matrix with eigenvalues $\lambda = 1, 2, 2$, then A is (always) not diagonalizable

(f) If $\hat{\mathbf{x}}$ is the orthogonal projection of $\mathbf{x}$ on W, then $\hat{\mathbf{x}}$ is orthogonal to $\mathbf{x}$.

(g) If $\hat{\mathbf{u}}$ is the orthogonal projection of $\mathbf{u}$ on $Span\,\{\mathbf{v}\}$, then:

$$\hat{\mathbf{u}} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{u}$$

(h) If Q is an orthogonal matrix, then Q is invertible.

2. *(15 points)* Label the following statements as **TRUE** or **FALSE**. In this question, you **HAVE** to justify your answer!!!

This means:

- If the answer is **TRUE**, you have to explain **WHY** it is true (possibly by citing a theorem)

- If the answer is **FALSE**, you have to give a specific **COUNTEREXAMPLE**. You also have to explain why the counterexample is in fact a counterexample to the statement!

IMPORTANT NOTE: If A is diagonalizable, explain why! And if A is not diagonalizable, *show* why it isn't!

(a) *(5 points)* If A is diagonalizable, then A is invertible.

(b) *(10 points, longer)* If A is invertible, then A is diagonalizable

3. *(10 points)* For the following matrix A, find a basis for $Nul(A)$, $Row(A)$, $Col(A)$, and find $Rank(A)$:

$$A = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 1 & 2 & -4 & 10 & 13 & -12 \\ 1 & -1 & -1 & 1 & 1 & -3 \\ 1 & -3 & 1 & -5 & -7 & 3 \\ 1 & -2 & 0 & 0 & -5 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4. *(10 points)* Let $\mathcal{B} = \{7 - 2t, 2 - t\}$, and $\mathcal{C} = \{4 + t, 5 + 2t\}$ be bases for P_1.

Calculate $[\mathbf{x}]_{\mathcal{C}}$ given $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Hint: First calculate a change-of-coordinates matrix!

5. *(10 points)* Define $T : M_{2\times 2} \to M_{2\times 2}$ by:

$$T(A) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} A$$

Find the matrix of T relative to the basis:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ of } M_{2\times 2}$$

6. *(30 points)* Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$, where:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Note: Show *all* your work!

7. *(4 points)* This question gives a proof of the Cauchy-Schwarz inequality:

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$$

(a) *(1 point)* What is the formula of $\hat{\mathbf{u}}$, the projection of $\mathbf{u}$ on $Span\,\{\mathbf{v}\}$?

(b) *(1 point)* Circle the correct answer:

(A) $\|\hat{\mathbf{u}}\| \leq \|\mathbf{u}\|$

(B) $\|\mathbf{u}\| \leq \|\hat{\mathbf{u}}\|$

(c) *(2 points)* Use your formula in (b) and your answer in (c) to solve for $\mathbf{u} \cdot \mathbf{v}$ and (hence) derive the Cauchy-Schwarz inequality!

Note: Be careful about when to put $|\cdot|$ or $\|\cdot\|$.

8. *(5 points)* Suppose $\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is orthonormal. Show that $\mathcal{B}$ is linearly independent!

Hint: Use hugging!

Note: Let me start the proof for you:

Suppose $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$.

Goal: Show that $a = b = c = 0$