MATH 54 – MIDTERM 2 – SOLUTIONS

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1. *(16 points, 2 points each)*

Label the following statements as **T** or **F**. **Write your answers in the box below!**

NOTE: In this question, you do **NOT** have to show your work! Don't spend *too* much time on each question!

(a) **TRUE** If A is diagonalizable, then A^3 is diagonalizable.

($A = PDP^{-1}$, so $A^3 = PD^3P = \widetilde{P}\widetilde{D}\widetilde{P}^{-1}$, where $\widetilde{P} = P$ and $\widetilde{D} = D^3$, which is diagonal)

(b) **TRUE** If A is a 3×3 matrix with 3 (linearly independent) eigenvectors, then A is diagonalizable

(This is one of the facts we talked about in lecture, the point is that to figure out if A is diagonalizable, look at the eigenvectors)

(c) **TRUE** If A is a 3×3 matrix with eigenvalues $\lambda = 1, 2, 3$, then A is invertible

(No eigenvalue which is 0, so by the IMT, A is invertible)

(d) **TRUE** If A is a 3×3 matrix with eigenvalues $\lambda = 1, 2, 3$, then A is (always) diagonalizable

(this is the useful test we've been talking about in lecture, A is diagonalizable since it has 3 distinct eigenvalues)

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(e) **FALSE** If A is a 3×3 matrix with eigenvalues $\lambda = 1, 2, 2$, then A is (always) not diagonalizable

(Take $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, it is diagonal, hence diagonalizable)

(f) **FALSE** If $\hat{\mathbf{x}}$ is the orthogonal projection of $\mathbf{x}$ on W, then $\hat{\mathbf{x}}$ is orthogonal to $\mathbf{x}$.
(Draw a picture)

(g) **FALSE** If $\hat{\mathbf{u}}$ is the orthogonal projection of $\mathbf{u}$ on $Span\{\mathbf{v}\}$, then:

$$\hat{\mathbf{u}} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{u}$$

(It's $\hat{\mathbf{u}} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v}$, it has to be a multiple of $\mathbf{v}$)

(h) **FALSE** If Q is an orthogonal matrix, then Q is invertible.
(Q might not be square!)

2. *(15 points)* Label the following statements as **TRUE** or **FALSE**. In this question, you **HAVE** to justify your answer!!!

(a) **FALSE** If A is diagonalizable, then it is invertible.

For example, take $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. It is diagonalizable **because it is diagonal**, but it is not invertible!

(b) **FALSE** If A is invertible, then A is diagonalizable

Take $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (this is the 'magic counterexample' we talked about in lecture). It is invertible because $det(A) = 1 \neq 0$. To show it is not diagonalizable, let's find the eigenvalues and eigenvectors of A:

Eigenvalues:

$$det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 = 0$$

Which gives us $\lambda = 1$.

Eigenvectors:

$$Nul(I - A) = Nul \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Which gives $-y = 0$, so $y = 0$, hence:

$$Nul(I - A) = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \right\} = Span \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

Since there is only one (linearly independent) eigenvector, A is not diagonalizable!

3. *(10 points)* For the following matrix A, find a basis for $Nul(A)$, $Row(A)$, $Col(A)$, and find $Rank(A)$:

$$A = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 1 & 2 & -4 & 10 & 13 & -12 \\ 1 & -1 & -1 & 1 & 1 & -3 \\ 1 & -3 & 1 & -5 & -7 & 3 \\ 1 & -2 & 0 & 0 & -5 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\underline{Nul(A)}$ Since the right-hand-side is not in reduced row-echelon form, let's further row-reduce it:

$$\begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 4 & 5 & -6 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 0 & 9 & 2 \\ 0 & 1 & -1 & 0 & 7 & 3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(I first subtracted the second row from the first, and then subtracted 3 times the third row from the second and 4 times the third row from the first)

Now if $A\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ t \\ s \\ r \end{bmatrix}$, then we get:

$$\begin{cases} x - 2z + 9s + 2r = 0 \\ y - z + 7s + 3r = 0 \\ t - s - 2r = 0 \end{cases}$$

That is:

$$\begin{cases} x = 2z - 9s - 2r \\ y = z - 7s - 3r \\ t = s + 2r \end{cases}$$

Hence we get:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ t \\ s \\ r \end{bmatrix} = \begin{bmatrix} 2z - 9s - 2r \\ z - 7s - 3r \\ z \\ s + 2r \\ s \\ r \end{bmatrix} = z \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

And therefore:

$$Nul(A) = Span \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\underline{Row(A)}$ Notice that there are pivots in the first, second, and third row, hence:

$$Row(A) = Span \left\{ \begin{bmatrix} 1 \\ 1 \\ -3 \\ 7 \\ 9 \\ -9 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 3 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ -2 \end{bmatrix} \right\}$$

$\underline{Col(A)}$ Notice that there are pivots in the first, second, and fourth columns, hence:

$$Col(A) = Span \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ 1 \\ -5 \\ 0 \end{bmatrix} \right\}$$

(Notice that you had to go back to the matrix A to find a basis for $Col(A)$)

$\underline{Rank(A)}$ There are 3 pivots, hence $Rank(A) = 3$.

4. *(10 points)* Let $\mathcal{B} = \{7-2t, 2-t\}$, and $\mathcal{C} = \{4+t, 5+2t\}$ be bases for P_1.

Calculate $[\mathbf{x}]_{\mathcal{C}}$ given $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Hint: First calculate a change-of-coordinates matrix!

First of all, identifying polynomials with their number codes, we get $\mathcal{B} = \left\{ \begin{bmatrix} 7 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$.

$$[\mathcal{C} \mid \mathcal{B}] = \begin{bmatrix} 4 & 5 & 7 & 2 \\ 1 & 2 & -2 & -1 \end{bmatrix} \to \begin{bmatrix} 4 & 5 & 7 & 2 \\ 0 & -3 & 15 & 6 \end{bmatrix}$$
$$\to \begin{bmatrix} 4 & 5 & 7 & 2 \\ 0 & 1 & -5 & 2 \end{bmatrix} \to \begin{bmatrix} 4 & 0 & 32 & 12 \\ 0 & 1 & -5 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 8 & 3 \\ 0 & 1 & -5 & -2 \end{bmatrix}$$

(first I added -4 times the second row to the first, then I divided the second row by -3, then I subtracted 5 times the second row from the first, and finally I divided the first row by 4)

Hence:

$$\mathcal{C} \overset{P}{\leftarrow} \mathcal{B} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}$$

We have:

$$[\mathbf{x}]_{\mathcal{C}} = \mathcal{C} \overset{P}{\leftarrow} \mathcal{B} \, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 20 \\ -13 \end{bmatrix}$$

5. *(10 points)* Define $T : M_{2\times 2} \to M_{2\times 2}$ by:

$$T(A) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} A$$

Find the matrix of T relative to the basis:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ of } M_{2\times 2}$$

$$T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

Hence the matrix of T is:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Note: If you wrote $\begin{bmatrix} 1 & 0 & 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}$, I took off 4 points! It's **very** important to convert the vectors you found into *column* vectors!!!

6. *(30 points)* Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$, where:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Eigenvalues:

$$\begin{aligned} det(\lambda I - A) = & \begin{vmatrix} \lambda - 1 & -1 & -1 \\ 1 & \lambda - 3 & -1 \\ 1 & -1 & \lambda - 3 \end{vmatrix} \\ = & (\lambda - 1) \begin{vmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{vmatrix} - (-1) \begin{vmatrix} 1 & -1 \\ 1 & \lambda - 3 \end{vmatrix} + (-1) \begin{vmatrix} 1 & \lambda - 3 \\ 1 & -1 \end{vmatrix} \\ = & (\lambda - 1)((\lambda - 3)^2 - 1) + (\lambda - 3) + 1 - (-1 - (\lambda - 3)) \\ = & (\lambda - 1)(\lambda^2 - 6\lambda + 9 - 1) + \lambda - 3 + 2 + \lambda - 3 \\ = & \lambda^3 - 6\lambda^2 + 8\lambda - \lambda^2 + 6\lambda - 8 + 2\lambda - 4 \\ = & \lambda^3 - 7\lambda^2 + 16\lambda - 12 \end{aligned}$$

Now by the rational roots theorem, the only numbers a which divide -12 are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$, and the only numbers b which divide -1 are ± 1, hence by the rational roots theorem we should try $\lambda = \frac{a}{b} = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$.

After trying some roots, you should get $\boxed{\lambda = 2}$ works! Now use long division:

$$\begin{array}{r} X^2 \quad - 5X \quad + 6 \\ X - 2 \overline{\big) \; X^3 - 7X^2 + 16X - 12} \\ \underline{- X^3 + 2X^2 \qquad\qquad\qquad} \\ - 5X^2 + 16X \qquad \\ \underline{5X^2 - 10X} \qquad \\ 6X - 12 \\ \underline{- 6X + 12} \\ 0 \end{array}$$

So $\lambda^3 - 7\lambda^2 + 16\lambda - 12 = (\lambda - 2)(\lambda^2 - 5\lambda + 6) = (\lambda - 2)(\lambda - 2)(\lambda - 3) = (\lambda - 2)^2(\lambda - 3)$

Hence the eigenvalues are $\boxed{\lambda = 2, 3}$

Eigenvectors:

$\underline{\lambda = 2}$:

$$Nul(2I - A) = Nul \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} = Nul \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

But then $x - y - z = 0$, so $x = y + z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y + z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence:

$$Nul(2I - A) = Span \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\underline{\lambda = 3}$:

$$Nul(3I - A) = Nul \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} = Nul \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix} = Nul \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

But then $x = z$ and $y = z$, so:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ z \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence:

$$Nul(3I - A) = Span\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$$

Answer:

$$D = \begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix}, P = \begin{bmatrix}1 & 1 & 1\\1 & 0 & 1\\0 & 1 & 1\end{bmatrix}$$

7. *(4 points)* This question gives a proof of the Cauchy-Schwarz inequality:

$$|\mathbf{u}\cdot\mathbf{v}| \leq \|\mathbf{u}\|\,\|\mathbf{v}\|$$

(a) *(1 point)* What is the formula of $\hat{\mathbf{u}}$, the projection of $\mathbf{u}$ on $Span\,\{\mathbf{v}\}$?

$$\hat{\mathbf{u}} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}}\cdot\mathbf{v}\right)\mathbf{v}$$

(b) *(1 point)* Circle the correct answer:

(A) $\boxed{\|\hat{\mathbf{u}}\| \leq \|\mathbf{u}\|}$

(B) $\|\mathbf{u}\| \leq \|\hat{\mathbf{u}}\|$
(draw a picture)

(c) *(2 points)* Use your formula in (b) and your answer in (c) to solve for $\mathbf{u}\cdot\mathbf{v}$ and (hence) derive the Cauchy-Schwarz inequality!

Note: Be careful about when to put $|\cdot|$ or $\|\cdot\|$.

First we use (c), then use (a), and finally take $\mathbf{u}\cdot\mathbf{v}$ outside of $\|\cdot\|$:

$$\begin{aligned}
\|\hat{\mathbf{u}}\| &\leq \|\mathbf{u}\| \\
\left\|\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}}\cdot\mathbf{v}\right)\mathbf{v}\right\| &\leq \|\mathbf{u}\| \\
\left|\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}}\cdot\mathbf{v}\right|\|\mathbf{v}\| &\leq \|\mathbf{u}\| \\
\frac{|\mathbf{u}\cdot\mathbf{v}|}{\|\mathbf{v}\|^{\not 2}}\not{\|\mathbf{v}\|} &\leq \|\mathbf{u}\| \\
\frac{|\mathbf{u}\cdot\mathbf{v}|}{\|\mathbf{v}\|} &\leq \|\mathbf{u}\| \\
|\mathbf{u}\cdot\mathbf{v}| &\leq \|\mathbf{u}\|\,\|\mathbf{v}\|
\end{aligned}$$

8. *(5 points)* Suppose $\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is orthonormal. Show that $\mathcal{B}$ is linearly independent!

Hint: Use hugging!

Note: Let me start the proof for you:

Suppose $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$.

Goal: Show that $a = b = c = 0$

First dot the above equation with $\mathbf{u}$ and use orthonormality:

$$\begin{aligned}(a\mathbf{u} + b\mathbf{v} + c\mathbf{w}) \cdot \mathbf{u} =& \mathbf{0} \cdot \mathbf{u} = 0 \\ a\mathbf{u} \cdot \mathbf{u} + b\mathbf{v} \cdot \mathbf{u} + c\mathbf{w} \cdot \mathbf{u} =& 0 \\ a(1) + b(0) + c(0) =& 0 \\ a =& 0\end{aligned}$$

Hence $b\mathbf{v} + c\mathbf{w} = \mathbf{0}$. Now dot this with $\mathbf{v}$ and use orthonormality:

$$\begin{aligned}(b\mathbf{v} + c\mathbf{w}) \cdot \mathbf{v} =& \mathbf{0} \cdot \mathbf{v} = 0 \\ b\mathbf{v} \cdot \mathbf{v} + c\mathbf{w} \cdot \mathbf{v} =& 0 \\ b(1) + c(0) =& 0 \\ b =& 0\end{aligned}$$

Hence $c\mathbf{w} = \mathbf{0}$. Finally, dot this with $\mathbf{w}$:

$$\begin{aligned}c\mathbf{w} \cdot \mathbf{w} =& \mathbf{0} \cdot \mathbf{w} = 0 \\ c(1) =& 0 \\ c =& 0\end{aligned}$$

Hence $a = b = c = 0$, and we're done!