1. 1) **ALWAYS TRUE** because.. A is diagonalizable implies that there is a basis consisting of eigenvectors. However, since λ is the only eigenvalue, it implies that there are n linearly independent eigenvectors in Nul $(A-\lambda I)$ so that $\dim \mathrm{Nul}\ (A-\lambda I) = n$. By the rank theorem, $A-\lambda I$ has its column space as the zero space so that $A-\lambda I = \mathbf{0}$. Therefore, $A=\lambda I$.

   2) **SOMETIMES FALSE** because.. we already know one good example of non-diagonalizable matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

   Also, we know that an upper triangular matrix with n distinct eigenvalues is diagonalizable. So, we can try, by being more tricky,

$$\begin{bmatrix} n & n & \cdots & n & n \\ 0 & n-1 & \cdots & n-1 & n-1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & 2 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & n-2 & n-2 \\ 0 & 0 & \cdots & 0 & n-1 \end{bmatrix} = \begin{bmatrix} n & n & \cdots & n & n \\ 0 & n & \cdots & n & n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & n & n \\ 0 & 0 & \cdots & 0 & n \end{bmatrix}$$

   Then, left two matrices respectively have n distinct eigenvalues so that they both are diagonalizable. However, as I mentioned just above, the matrix on the Right-hand side is not diagonalizable.

   3) **ALWAYS TRUE** because.. For the columns of A being orthonormal means that $A^TA = I$. In particular, because A is $n\times n$, $A^TA = I_n$. In such a case, we know that A is invertible so that A^T becomes the inverse of A, that is, $A^T = A^{-1}$. Now, we can say that $AA^T = I_n$ as well.[1]Using $A = (A^T)^T$, we can say

$$(A^T)^TA^T = I.$$

   Hence, again, the columns of A^T are orthonormal. However, since the columns of A^T is the rows of A, A's rows are orthonormal as well.

   4) **ALWAYS TRUE** because.. For any matrix (not necessariliy to be square) A and B, we have

$$\text{rank}AB \le \text{rank}A \text{ and } \text{rank}B.$$[2]

   Now, let $B = PAP^{-1}$ hold for some invertible P[3]. Then, rank$B$ =rankPAP^{-1} $\le$rankAP^{-1} $\le$rank$A$. However, $A = P^{-1}BP$ gives a similar result that rank$A$ =rank$P^{-1}BP$ $\le$rankBP $\le$rank$B$. Therefore, rank$A$ =rank$B$.[4]

   5) **ALWAYS TRUE** because.. in general, subtracting $\mathbf{v}_3$'s projection from $\mathbf{v}_3$, we get an orthogonal vector to the projecting space. So, in this problem,

   $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ form an orthogonal set
   if and only if $\mathbf{v}_3$ is orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$
   if and only if $\mathbf{v}_3$ is orthogonal to the span of $\mathbf{v}_1$ and $\mathbf{v}_2$
   if and only if the projection of $\mathbf{v}_3$ is the zero vector.

[1] Note that this argument is only available for a square matrix A.
[2] You can look up the last page for the proof. It is READABLE and RECOMMENDED to read.
[3] Definition of SIMILARITY
[4] As you can easily figure out, rank$A$ = rank$B$ if $A = PBQ$ for some invertible P and invertible Q.

2. 1) **| a, b, c |** A is not invertible $\iff \det A = 0 \iff \det(A - 0 \cdot I) = 0 \iff \det(I_n - A - 1 \cdot I_n) = 0$. So, a) and b) are correct answers. c) is obviously correct from a). However, d) is never true because it says that 0 can never be an eigenvalue. e) Diagonalizability has nothing to do with invertibility. (Also, we have an easy counterexample $A = I_n$.)

   2) **| a, d, e |** Let's first take a look at the characteristic equations of each matrices.

      a) gives $(\lambda - 1)^2 - 9 = \lambda^2 - 2\lambda - 8 = 0$. It has two distinct roots 4 and -2. So, diagonalizable.
      b) gives $(\lambda - 1)^2 + 9$. This has no real solutions. So, not our business.
      c) gives $(1 - \lambda)^3$. Just like problems 1-1), the only case it is diagonalizable is when $A = I_3$. Not correct.
      d) gives $((-1) - \lambda)(0 - \lambda)(1 - \lambda)$. It has three distinc real roots. So, correct.
      e) gives $(1-\lambda)(\lambda^2-1)$ (cofactor expansion using second column). So, need to check if $\dim$ Nul $(A-I) = 2$ or not. It is, in fact, 2. So, correct.

   3) **| a, b |** All of them has the same dimension for domain and codomain. Because all of them are linear, we only need to check if the kernels are zero or not.

      a) If $p''(0) = 0$, then $p(x) = ax + b$ for some a and b. Moreover, given $p'(0) = 0$, we know $a = 0$. Finally, $p(0) = 0$ gives $b = 0$. So, $p(x) = 0$ if $\begin{bmatrix} p(0) \\ p'(0) \\ p''(0) \end{bmatrix} = \mathbf{0}$. So, it is an isomorphism.
      b) If, for $p(x) = ax^2 + bx + c$, $p(0) = 0$, $p(1) = 0$, $p(2) = 0$, then $c = 0$, $a + b + c = 0$, $4a + 2b + c = 0$ so that $a = b = c = 0$ so that $p(x) = 0$. So, the kernel is the zero space. It is an isomorphism.
      c) The kernel of this map consists of $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $a - b = 0$, $b - c = 0$, $c - a = 0$ so that $a = b = c$. The kernel is not the zero space, so it is not an isomorphism.
      d) Almost same argument as above : $a + b + c = 0$, $b + c = 0$, $a = 0$ does not imply that $a = b = c = 0$ ($\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ could be one counterexample for that.) So, the kernel is not the zero space, so it is not an isomorphism.
      e) Given $p(x) = ax^2 + bx + c$, $p'(x) + x^2 p''(x) = 2ax + b + 2ax^2$. So, the kernel's elements should satisfy $2a = 0$, $2a = 0$, $b = 0$. However, no restrictions for c exist. So, the kernel is not the zero space. Not an isomorphism.

   4) **| b, d |** Only need to check if vectors are of length 1 and mutually orthogonal (or the dot product of two different vectors is zero always). So, b), and d) are only correct answers.

   5) **| a, d, e |** First, note that the columns of A are orthogonal if and only if $A^T A$ is a diagonal matrix. In this sense, d) and e) are correct answers. However, for b) and c), there are no reasons. In fact,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

   gives counterexamples for both b) and c). The hardest one is a). One fact you should keep in mind is that (except for zero vectors) orthogonal vectors are linearly independent. So, if you row reduce A, then except for the columns which are the zero vectors, the other columns should become ones of standard basis vectors e_i's. However, the zero vectors with standard basis vectors always form an orthogonal set obviously. So, a) is also correct.

3. a) Let's say $Av = \lambda v$ and

$$v = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}.$$

Then, by taking the sum of all entries of the left side vector Av and then comparing with the sum of all entries of the right side vector λv, we know that $0 = \lambda(x+y+z+w)$[5]. So, $\lambda = 0$ or $x+y+z+w$.

Next, $Ae_1 = -Ae_2$, $Ae_3 = -Ae_4$ and Ae_1, Ae_3 are linearly independent. So, the rank of A is 2. So, the eigenspace (assoc. with $\lambda = 0$) is of dimension 2. Now, we only need up to 2.

Let's go back to $\lambda = 0$ or $x+y+z+w = 0$. In the case when $\lambda \neq 0$, we have $x+y+z+w = 0$ and also, by comparing the last entry of Av, which is 0, with the last entry of λv, which is λw, we can easily figure out $w = 0$. So, we have $x+y+z = 0$. However, $x+y+z = 0$ is the subspace of $\mathbb{R}^4$ with dimension 2. This space, HOPEFULLY, is the subspace of dimension 2 that we are looking for.

Now, if v is an eigenvector associated with nonzero λ, we know that $w = 0$ and $x+y+z = 0$. Let $z = -x-y$. Then, $Av = \begin{bmatrix} 2x \\ -y \\ -2x+y \\ 0 \end{bmatrix}$. By thinking about the first entry, we know that 2 might be an eigenvalue or $x = 0$. However, if $\lambda = 2$, then $y = 0$ gives an eigenvector indeed. Next, comparing the second entries, we get that 1 might be an eigenvalue or $y = 0$. Actually, this $\lambda = 1$ is an eigenvalue.

So, now we have 2-dimensional eigenspace of $\lambda = 0$ and 2 1-dimensional eigenspaces of (respectively) $\lambda = -1$, $\lambda = 2$. So,

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

b) Set $\mathbf{x} = P\mathbf{y}$ for some $\mathbf{y}$. Then, $A^{54}\mathbf{x} = A^{54}P\mathbf{y} = PD^{54}\mathbf{y}$. We want this to be $P\mathbf{y}$. So, basically we need $\mathbf{y}$ such that $D^{54}\mathbf{y} = \mathbf{y}$. Note that $D^{54} = \text{diag}(0,0,2^{54},1)$ ($\text{diag}(a,b,c,d)$ means a diagonal matrix with its diagonal entries as a, b, c, and d). So, we have

$$\mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ so that } \mathbf{x} = P\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

[5] Observe that the sums of entries in each columns are zero

4. a) What we need is a basis $\{b_1, b_2\}$ for $\mathbb{P}_1$ and a basis $\{c_1, c_2, c_3\}$ for $\mathbb{P}_2$ such that $T(b_1) = c_1$, $T(b_2) = c_2$. So, just pick any basis for $\mathbb{P}_1$. I would choose $\{1, x\}$ cause its the standard one. Then, $c_1 = T(b_1) = T(1) = -x$ and $c_2 = T(b_2) = T(x) = 1 - x^2$. For c_3, we can choose arbitrary vector in $\mathbb{P}_2$ as long as c_3 is linearly independent with c_1 and c_2. So, $\mathcal{B} = \{1, x\}$ and $\mathcal{C} = \{-x, 1 - x^2, x^2\}$ could be one example of such bases.

   b) Suppose that there are such bases. Then, the range of T (which is the span of $T(b_1)$ and $T(b_2)$) will be the span of $c_1 - c_2$ and $c_2 - c_1$. However, $c_2 - c_1 = -(c_1 - c_2)$, so the range of T is spanned by only one vector $c_1 - c_2$ so that the dimension of range$T = 1$. But, in a), we already showed that $\dim$ range$T = 2$, this is impossible. Hence, there are no such bases.

5. a) We need to find the projection of $\mathbf{v}$ on to Span$\{\mathbf{u}\}$ and then subtract it from $\mathbf{v}$ so that the orthogonal part only remains.

$$\mathbf{w} = \mathbf{v} - \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \frac{-1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \end{bmatrix}.$$

   b) By **Best Approximation Theorem**, we only need to find the orthogonal projection of the given vector on to W. To avoid any confusion, let's denote the vector by $\mathbf{x}$. So, the orthogonal projection of $\mathbf{x}$ on to W is

$$\frac{\mathbf{u} \cdot \mathbf{x}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} + \frac{\mathbf{w} \cdot \mathbf{x}}{\mathbf{w} \cdot \mathbf{w}}\mathbf{w} = \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix}.$$

6. a) **SOMETIMES FALSE** because.. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ gives a counterexample for the statement.

   b) **ALWAYS TRUE** because.. $A = PDP^{-1}$ means $AP = PD$, not let v_i be the ith column of P. Then, the ith column of AP is Av_i. On the other hand, PD's ith column is $P \cdot \lambda_i \mathbf{e}_i = \lambda_i v_i$. So, v_i is an eigenvector. (In fact, you should mention that it is nonzero because the zero vector is always not an eigenvector.)

Proof of the inequality about rank

The column space of AB consists of all vectors of the form $AB\mathbf{x}$ and $AB\mathbf{x} = A(B\mathbf{x})$ is in the column space of A obviously. Hence, Col AB is always a subspace of Col A so that rank$AB \leq$ rank$A$. Now, given two matrices C and D, let's define $A = D^T$ and $B = C^T$. And then apply what we just get.

$$\text{rank}AB \leq \text{rank}A \text{ is the same as, in terms of } C \text{ and } D,\ \text{rank}D^T C^T \leq \text{rank}D^T.$$

However, we know that rank$A$ =rankA^T for any matrix A, so the above inequality becomes

$$\text{rank}CD \leq \text{rank}D.$$

Now, we can just regard C and D as new A and B. Hence, we can conclude that

$$\text{rank}AB \leq \text{rank}A \text{ and } \text{rank}AB \leq \text{rank}B.$$