Practice Midterm 2

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Problem	Score
1	/♡
2	/♡
3	/♡
4	/♡
5	/♡
6	/♡
Total	/6♡

Problem 1

Decide if the following statements are *always true* or *sometimes false*. JUSTIFY YOUR ANSWER.

a) Every orthogonal set is a linearly independent set.

FALSE when you have the zero space $\{\mathbf{0}\}$.

b) Two diagonalizable matrices A and B are similar if they have the same eigenvalues, counting multiplicities.

TRUE because A's diagonalization is similar to B's diagonalization. Note that they are just differ by some permutation of diagonal entries.

c) If A^3 is diagonalizable, then A is diagonalizable as well.

FALSE when $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

d) If A^3 is diagonalizable, then there exists diagonalizable B such that $A^3 = B^3$.

TRUE because $A^3 = PDP^{-1}$ for some invertible P and diagonal D. In particular, if $D = \text{diag}(\lambda_1, \cdots, \lambda_n)$, set $D_{1/3} = \text{diag}(\lambda_1^{1/3}, \cdots, \lambda_n^{1/3})$. Then, $D_{1/3}^3 = D$ so that B defined by $PD_{1/3}P^{-1}$ satisfies $A^3 = B^3$.

e) Let A be a $n \times n$ matrix. If the sum of entries in a column is zero for each column, then 0 is an eigenvalue of A.

TRUE because every column then lives in $x_1 + \cdots + x_n = 0$, which is an $(n-1)$ dimensional space. The number of columns is n, so they should be linearly dependent. So, A is not invertible and 0 is an eigenvalue.

f) Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_n$ are vectors in $\mathbb{R}^n$. If $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is an orthonormal set, then it is a basis for $\mathbb{R}^n$.

TRUE because an orthonormal set is a linearly independent set and there are n vectors in $\mathbb{R}^n$ (n-dimensional space).

g) If A and B are $n \times n$ invertible matrices, then AB is similar to BA.

TRUE because $AB = ABAA^{-1}$.

Problem 2

Define a linear transformation T from $\mathbb{P}_2$ to $\mathbb{P}_2$ as follows.

$$T(p(t)) = 3p(t) - tp'(t).$$

a) Let $\mathcal{E}$ be the standard basis for $\mathbb{P}_2$. Find the $\mathcal{E}$-matrix for T.

$T(1) = 3$, $T(t) = 2t$, $T(t^2) = t^2$, so the matrix is

$$[T]_{\mathcal{E}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

b) Is it possible to find a basis $\mathcal{B}$ for $\mathbb{P}_2$ such that

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}?$$

No since then T's eigenvalues will be $1, 1, 1$. However, we have already seen that T in fact have eigenvalues 1, 2, and 3.

Problem 3

Let A be

$$\begin{bmatrix} 3 & -4 & -4 \\ 2 & 1 & -4 \\ -2 & 0 & 5 \end{bmatrix}$$

whose characteristic polynomial $\chi_A(\lambda)$ is $-(\lambda-1)(\lambda-3)(\lambda-5)$.

a) Find 3 linearly independent eigenvectors and, using them, find a diagonal matrix D and an invertible matrix P such that

$$P^{-1}AP = D.$$

As usual, you need to find the null spaces of $A - I$, $A - 3I$, and $A - 5I$. In fact,

$$\text{Nul}(A-I) = \text{Span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{Nul}(A-3I) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\text{Nul}(A-5I) = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

So, $P = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ works for $P^{-1}AP = D$.

b) Find all possible D's. For each D, find one corresponding invertible matrix P such that $P^{-1}AP = D$.

Because P's columns are always eigenvectors, D's entries also should be all zero but eigenvalues on diagonal. So, possible D's are the matrices : diag$(1,3,5)$, diag$(1,5,3)$, diag$(3,1,5)$, diag$(3,5,1)$, diag$(5,1,3)$, and diag$(5,3,1)$. Corresponding P's could be matrices obtained by changing positions of columns.

Problem 4

1) Let T be a linear transformation from V to W. For bases $\mathcal{B}$ of V and $\mathcal{C}$ of W, let the matrix for T relative to $\mathcal{B}$ and $\mathcal{C}$ be
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$
Which of the following matrices could be a matrix for T (possibly, choosing different $\mathcal{B}'$ and $\mathcal{C}'$ from $\mathcal{B}$ and $\mathcal{C}$)?

a) $\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ e) $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

Basically, the matrix given above, as a linear transformation, has 2-dimensional range, so all the matrices having 2-dimensional range can be a matrix for T with an appropriate choice of $\mathcal{B}$ and $\mathcal{C}$. The answer is **a)**, **b)**, **c)**, **e)**.

2) Which of the following matrices are similar to
$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}?$$

a) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$ e) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Two similar matrices should have the same eigenvalues, counting multiplicities. So, only possibilities are c), d), e). However, the given matrix is not diagonalizable since the eigenspace associated with $\lambda = 1$ has dimension 1. But, d), e) are diagonalizable. However, a diagonalizable matrix is never similar to a non-diagonalizable matrix. So, only c) is possible. And, in fact, $P = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ gives $P^{-1}AP = B$ where A is the given matrix and B is c). The answer is **c)**.

3) Which of the following sets are orthogonal?

a) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ b) $\left\{ \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \end{bmatrix} \right\}$ c) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

d) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ e) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ -11 \\ 6 \end{bmatrix} \right\}$

The answer is **a)**, **b)**, **c)**.

Problem 5

Consider

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

Note that they are orthogonal to each other and let W be the span of $\{\mathbf{u}, \mathbf{v}\}$.

a) Define a linear transformation T from $\mathbb{R}^4$ to $\mathbb{R}^4$ as the orthogonal projection

$$T(\mathbf{x}) = \text{proj}_W(\mathbf{x}) = \frac{\mathbf{u} \cdot \mathbf{x}}{3}\mathbf{u} + \frac{\mathbf{v} \cdot \mathbf{x}}{3}\mathbf{v}.$$

Let's denote the $\mathcal{E}$-matrix of T by $[T]$. ($\mathcal{E}$ is the standard basis for $\mathbb{R}^4$.) Find eigenvalues of $[T]$.

$$T(e_1) = \begin{bmatrix} 2/3 \\ 1/3 \\ 0 \\ 1/3 \end{bmatrix} \qquad T(e_2) = \begin{bmatrix} 1/3 \\ 1/3 \\ -1/3 \\ 0 \end{bmatrix} \qquad T(e_3) = \begin{bmatrix} 0 \\ -1/3 \\ 2/3 \\ 1/3 \end{bmatrix} \qquad T(e_4) = \begin{bmatrix} 1/3 \\ 0 \\ 1/3 \\ 1/3 \end{bmatrix}$$

So,

$$[T] = \begin{bmatrix} 2/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & -1/3 & 0 \\ 0 & -1/3 & 2/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \end{bmatrix}.$$

Let $T(\mathbf{x}) = \lambda\mathbf{x}$ for some $\lambda \in \mathbb{R}$ and nonzero $\mathbf{x} \in \mathbb{R}^4$. Then, $\text{proj}_W(\mathbf{x}) = \lambda\mathbf{x}$. Recall that $\mathbf{x} = (\mathbf{x} - \text{proj}_W(\mathbf{x})) + \text{proj}_W(\mathbf{x})$ and $\mathbf{x} - \text{proj}_W(\mathbf{x}) \perp \text{proj}_W(\mathbf{x})$. Unless $\lambda = 0$, $\mathbf{x} - \text{proj}_W(\mathbf{x}) \perp \mathbf{x}$ by multiplying $1/\lambda$ to $\lambda\mathbf{x}$. So, $\mathbf{x} - \text{proj}_W(\mathbf{x}) \perp \mathbf{x} - \text{proj}_W(\mathbf{x})$ so that $\mathbf{x} - \text{proj}_W(\mathbf{x}) = 0$. In such a case $\mathbf{x} \in W$ already, so $T(\mathbf{x}) = \mathbf{x}$ so that $\lambda = 1$. So, eigenvalues are 1 and 0.

b) Is the matrix $[T]$ diagonalizable?

Yes, because Nul$[T]$ contains $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ and Nul$([T] - I)$ contains $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$. So, there are 4 linearly independent eigenvectors so that $[T]$ is diagonalizable. In particular, it is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Problem 6[1]

Let W be a subspace of $\mathbb{R}^n$. Given an orthogonal basis $\mathcal{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_m\}$ for W, recall that the formula of the orthogonal projection of $v \in \mathbb{R}^n$ onto W is given by

$$\frac{\mathbf{b}_1 \cdot v}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 + \cdots + \frac{\mathbf{b}_m \cdot v}{\mathbf{b}_m \cdot \mathbf{b}_m}\mathbf{b}_m.$$

Let's denote this by $\text{proj}_{W,\mathcal{B}}(v)$.[2]

a) Show that $v - \text{proj}_{W,\mathcal{B}}(v)$ is orthogonal to $\text{proj}_{W,\mathcal{B}}(v)$. Also, prove that $v - \text{proj}_{W,\mathcal{B}}(v) \in W^{\perp}$.[3]

Let's first check $v - \text{proj}_{W,\mathcal{B}}(v)$ is orthogonal to each of $\mathbf{b}_i$'s.

$$(v - \text{proj}_{W,\mathcal{B}}(v)) \cdot \mathbf{b}_i = v \cdot \mathbf{b}_i - \text{proj}_{W,\mathcal{B}}(v) \cdot \mathbf{b}_i.$$

However, $\text{proj}_{W,\mathcal{B}}(v) \cdot \mathbf{b}_i$ is $\frac{\mathbf{b}_i \cdot v}{\mathbf{b}_i \cdot \mathbf{b}_i}\mathbf{b}_i \cdot \mathbf{b}_i$ because only ith term is effective since $\mathbf{b}_j$'s are orthogonal to each other. So, $v - \text{proj}_{W,\mathcal{B}}(v)$ is orthogonal to each $\mathbf{b}_i$'s. So is to any linear combination of them so that is to W. Note that $\text{proj}_{W,\mathcal{B}}(v)$ is in W. Hence, we get the results.

[1]This problem is designed to prove that the formula for the orthogonal projection,

$$\frac{\mathbf{b}_1 \cdot v}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 + \cdots + \frac{\mathbf{b}_m \cdot v}{\mathbf{b}_m \cdot \mathbf{b}_m}\mathbf{b}_m,$$

is independent of the choice of an orthogonal basis $\{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_m\}$ for W.

[2]I intentionally put $\mathcal{B}$ to emphasize that this is the projection using the basis $\mathcal{B}$.

[3]Hint. Use the linearity property of an innder product $\cdots$ and the definition of *orthogonality*. In order to prove $v - \text{proj}_{W,\mathcal{B}} \in W^{\perp}$, you only need to show that $v - \text{proj}_{W,\mathcal{B}}$ is orthogonal to $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_m$.

b) Let $\mathcal{C} = \{\mathbf{c}_1, \cdots, \mathbf{c}_m\}$ be another orthogonal basis for W.[4] Prove that[5]

$$\text{proj}_{W,\mathcal{B}}(v) - \text{proj}_{W,\mathcal{C}}(v) \in W^{\perp}.$$

Note that $v - \text{proj}_{W,\mathcal{B}}(v) \in W^{\perp}$ by a). With the same argument, we have $v - \text{proj}_{W,\mathcal{C}}(v) \in W^{\perp}$. However, $W^{\perp}$ is a vector space, so

$$(v - \text{proj}_{W,\mathcal{B}}(v)) + (-1)(v - \text{proj}_{W,\mathcal{C}}(v)) \in W^{\perp}.$$

c) Assume that there is no nonzero vector v such that $v \in W$ and $v \in W^{\perp}$ at the same time, without a proof. Using this fact, prove that

$$\text{proj}_{W,\mathcal{B}}(v) - \text{proj}_{W,\mathcal{C}}(v) = 0$$

By definition, $\text{proj}_{W,\mathcal{B}}(v) \in W$ and so is $\text{proj}_{W,\mathcal{C}}(v)$. Because W is a subspace (so, a vector space), we have

$$\text{proj}_{W,\mathcal{B}}(v) - \text{proj}_{W,\mathcal{C}}(v) \in W.$$

Combining with the result of b), we get $\text{proj}_{W,\mathcal{B}}(v) - \text{proj}_{W,\mathcal{C}}(v) \in W$ and $\in W^{\perp}$ at the same time. So, $\text{proj}_{W,\mathcal{B}}(v) - \text{proj}_{W,\mathcal{C}}(v) = 0$ by the fact that $v \in W$ and $v \in W^{\perp}$ implies $v = 0$.

Therefore,

$$\text{proj}_{W,\mathcal{B}}(v) = \text{proj}_{W,\mathcal{C}}(v).$$

So, we can conclude that the formula of the orthogonal projection does not depend on the choice of an orthogonal basis.

Remark. Why does $v \in W$ and $v \in W^{\perp}$ at the same time imply $v = 0$?

If then, $v \cdot v = 0$ because $v \in W$ and $v \in W^{\perp}$. However, $||v||^2 = 0$ implies $v = 0$.

[4] From a), we have $v - \text{proj}_{W,\mathcal{C}} \in W^{\perp}$.

[5] Hint. $W^{\perp}$ is a subspace of $\mathbb{R}^n$ (you can use this fact without a proof) so that $W^{\perp}$ is closed under addition and scalar multiplication.