MATH 54 – MIDTERM 1

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Name: ______________________________________

Note: The reason this exam is so long is because it combines the first midterm and half of the second midterm that I gave during Summer 2012.

1		18
2		30
3		15
4		20
5		15
6		15
7		15
8		10
9		5
10		5
11		2
Total		150

Date: Friday, June 29th, 2012.

1. *(18 points, 2 pts each)*

Label the following statements as **TRUE (T)** or **FALSE (F)**.

NOTE: In this question, you do **NOT** have to justify any answers! Also, don't spend *too* much time on each question!

(a) If the **augmented** matrix of the system $A\mathbf{x} = \mathbf{b}$ has a pivot in the last column, then the system $A\mathbf{x} = \mathbf{b}$ has no solution.

(b) If A and B are invertible 2×2 matrices, then $(AB)^{-1} = A^{-1}B^{-1}$

(c) If A is a 3×3 matrix such that the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^3$.

(d) The general solution to $A\mathbf{x} = \mathbf{b}$ is of the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_0$, where $\mathbf{x}_p$ is a *particular* solution to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x}_0$ is the *general* solution to $A\mathbf{x} = \mathbf{0}$.

(e) If P and D are $n \times n$ matrices, then $det(PDP^{-1}) = det(D)$

(f) If A is a $m \times n$ matrix, then $dim(Nul(A)) + Rank(A) = m$

(g) If $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$, then $Nul(T) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

(h) The set of polynomials $\mathbf{p}$ in P_2 such that $\mathbf{p}(3) = 0$ is a subspace of P_2

(i) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$

2. *(30 points, 5 points each)* Label the following statements as **TRUE** or **FALSE**. In this question, you **HAVE** to justify your answer!!!

   This means:

   - If the answer is **TRUE**, you have to explain **WHY** it is true (possibly by citing a theorem)

   - If the answer is **FALSE**, you have to give a specific **COUNTEREXAMPLE**. You also have to explain why the counterexample is in fact a counterexample to the statement!

   (a) If A and B are any 2×2 matrices, then $AB = BA$

   (b) The matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 3 & 0 \end{bmatrix}$ is not invertible.

   (c) The set of matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ is a subspace of $M_{2\times 2}$.

(d) The matrix of the linear transformation T which reflects points in $\mathbb{R}^2$ about the x-axis and then about the y-axis is the same as the matrix of the linear transformation S which rotates points in $\mathbb{R}^2$ about the origin by 180 degrees counterclockwise.

(e) The following set is a basis for P_2: $\{1, 1+t, 1+t+t^2\}$

(f) If V is a set that contains the $\mathbf{0}$-vector, and such that whenever $\mathbf{u}$ and $\mathbf{v}$ are in V, then $\mathbf{u}+\mathbf{v}$ is in V, then V is a vector space!

3. *(15 points)* Solve the following system of equations (or say it has no solutions):

$$\begin{cases} 2x + 2y + z = 2 \\ 3x + 4y + 2z = 3 \\ x + 2y - z = -3 \end{cases}$$

4. *(20 points)* Solve the following system $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{bmatrix} 1 & 1 & 1 & -3 \\ 2 & 3 & 1 & -6 \\ -1 & 2 & -4 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 8 \\ 3 \end{bmatrix}$$

Write your answer in (parametric) vector form

5. *(15 points, 5 points each)*

(a) Calculate AB, or say that AB is undefined.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

(b) Calculate AB, or say that AB is undefined.

$$A = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 3 & 0 \end{bmatrix}$$

(c) Calculate A^2, where:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

6. *(15 points)* Find A^{-1} (or say 'A is not invertible') where:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

7. *(15 points)* Find $det(A)$, where:

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 2 & 0 & 4 & 0 & 5 \\ 1 & 2 & 5 & -2 & 0 \\ 2 & 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

8. *(10 points)* For the following matrix A, find a basis for $Nul(A)$, $Col(A)$, and find $Rank(A)$:

$$A = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 1 & 2 & -4 & 10 & 13 & -12 \\ 1 & -1 & -1 & 1 & 1 & -3 \\ 1 & -3 & 1 & -5 & -7 & 3 \\ 1 & -2 & 0 & 0 & -5 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

9. *(5 points)* Define $T : C^{\infty}(\mathbb{R}) \to C^{\infty}(\mathbb{R})$ (the space of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$) by:

$$T(y) = y'' - 5y' + 6y$$

Show that T is a linear transformation.

10. *(5 points)*

(a) *(1 point)* If $T : V \to W$ is a one-to-one linear transformation and $T(\mathbf{x}) = \mathbf{0}$, what can you say about $\mathbf{x}$?

(b) *(4 points)* Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly *independent* vectors (in V) and $T : V \to W$ is a *one-to-one* linear transformation. Show that $T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{w})$ are also linearly independent.

Hint: Use (a)!!!

Let me start the proof for you:

Suppose $aT(\mathbf{u}) + bT(\mathbf{v}) + cT(\mathbf{w}) = \mathbf{0}$.

We want to show that $a = b = c = 0$

Note: You can actually use this problem to show that if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for $\mathbb{R}^3$ and A is an invertible matrix, then $\{A\mathbf{u}, A\mathbf{v}, A\mathbf{w}\}$ is also a basis for $\mathbb{R}^3$. This provides a neat way of creating new bases for $\mathbb{R}^3$: Start with any old basis $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ for $\mathbb{R}^3$, then $\{A\mathbf{u}, A\mathbf{v}, A\mathbf{w}\}$ is a (usually) a new basis for $\mathbb{R}^3$.

12. *(2 points)* Find $det(A)$, where:

$$A = \begin{bmatrix} 1 & x & x^2 & x^3 \\ 1 & y & y^2 & y^3 \\ 1 & z & z^2 & z^3 \\ 1 & t & t^2 & t^3 \end{bmatrix}$$

where x, y, z, t are distinct real numbers. This is called a **Vandermonde** matrix!

Hint: Calculating this directly is going to drive you nuts! Can you think about another way of calculating the determinant?

Note: You might find the formula $p^3 - q^3 = (p-q)(p^2+pq+q^2)$ (where p and q are real numbers) useful!