MATH 54 – MIDTERM 1 – SOLUTIONS

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1. *(18 points, 2 pts each)*

Label the following statements as **TRUE (T)** or **FALSE (F)**.

(a) **TRUE** If the **augmented** matrix of the system $A\mathbf{x} = \mathbf{b}$ has a pivot in the last column, then the system $A\mathbf{x} = \mathbf{b}$ has no solution.

(that's because there's a row of the form $\begin{bmatrix} 0 & 0 & \cdots & 0 \; b \end{bmatrix}$, where $b \neq 0$)

(b) **FALSE** If A and B are invertible 2×2 matrices, then $(AB)^{-1} = A^{-1}B^{-1}$

(it's $(AB)^{-1} = B^{-1}A^{-1}$, *reverse order*)

(c) **TRUE** If A is a 3×3 matrix such that the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^3$.

(the IMT implies that A is invertible, and the IMT again implies the desired result)

(d) **TRUE** The general solution to $A\mathbf{x} = \mathbf{b}$ is of the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_0$, where $\mathbf{x}_p$ is a *particular* solution to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x}_0$ is the *general* solution to $A\mathbf{x} = \mathbf{0}$.

(See section 1.5)

(e) **TRUE** If P and D are $n \times n$ matrices, then $det(PDP^{-1}) = det(D)$

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$$det(PDP^{-1}) = det(P)det(D)det(P^{-1}) = \cancel{det(P)}det(D)\frac{1}{\cancel{det(P)}} = det(D)$$

(f) **FALSE** If A is a $m \times n$ matrix, then $dim(Nul(A)) + Rank(A) = m$

(it's $dim(Nul(A)) + Rank(A) = n$, not m)

(g) **FALSE** If $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$, then $Nul(T) = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

$$Nul(T) = \left\{\begin{bmatrix} x \\ y \end{bmatrix} \text{ s.t. } \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\} = \left\{\begin{bmatrix} x \\ y \end{bmatrix} \text{s.t.} x = 0\right\} = \left\{\begin{bmatrix} 0 \\ y \end{bmatrix}\right\} = Span\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

(h) **TRUE** The set of polynomials $\mathbf{p}$ in P_2 such that $\mathbf{p}(3) = 0$ is a subspace of P_2

(You can easily check that the 0-polynomial is in it, that it is closed under addition and scalar multiplication)

(i) **FALSE** $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$ (it's not even a *subset* of $\mathbb{R}^3$!!!)

2. *(30 points, 5 points each)* Label the following statements as **TRUE** or **FALSE**. In this question, you **HAVE** to justify your answer!!!

(a) **FALSE** If A and B are any 2×2 matrices, then $AB = BA$

Take for example, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Then:

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad BA = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$$

which are not equal to each other!

(in fact, almost any two matrices you chose will give you a counterexample! The most important thing is that you had to find explicit A and B and you had to show that $AB \neq BA$)

(b) **TRUE** The matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 3 & 0 \end{bmatrix}$ is not invertible.

Notice that the first and the third column of the matrix are equal, hence the columns of A are linearly dependent, so by the IMT A is not invertible!

Note: Many many other answers were possible! For example, you could calculate $det(A) = 0$, or you could row-reduce and say that the matrix has only 2 pivots. Any of those answers is acceptable!

(c) **TRUE** The set of matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ is a subspace of $M_{2\times 2}$.

If you denote that set by V, then you get:

$$V = Span\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

And since the span of anything is a vector space, V is a vector space, and hence a subspace of $M_{2\times 2}$.

Alternatively you could have shown in the usual way that the O matrix is in it, and that it is closed under addition and scalar multiplication.

(d) **TRUE** The matrix of the linear transformation T which reflects points about the x-axis and then about the y-axis is the same as the matrix of the linear transformation S which rotates points about the origin by 180 degrees counterclockwise.

Calculate $T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$ and $T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix}$

Hence the matrix of T is $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$

Calculate $S\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$ and $S\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix}$

Hence the matrix of S is $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$.

And notice the two matrices are the same!

(e) **TRUE** The following set is a basis for P_2: $\{1, 1+t, 1+t+t^2\}$.

Linear independence: Suppose $a(1)+b(1+t)+c(1+t+t^2) = 0$, then $(a+b+c)+(b+c)t+ct^2 = 0$, hence $c = 0$, hence $b = 0$, hence $a = 0$, hence $a = b = c = 0$, and the polynomials are linearly independent.

Span: Since P_2 is 3-dimensional, and the set contains 3 elements, hence the set also spans P_2

Therefore the set is a basis for P_2.

Note: There were many, many, many other ways to show why this is true! One way is to consider the matrix $\begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$ and notice its determinant is $1 \neq 0$, hence it is invertible, hence its columns are linearly independent and span $\mathbb{R}^3$. Or you could use the Wronskian (if you want to make me happy :)).

(f) **FALSE** If V is a set that contains the $\mathbf{0}$-vector, and such that whenever $\mathbf{u}$ and $\mathbf{v}$ are in V, then $\mathbf{u}+\mathbf{v}$ is in V, then V is a vector space!

Consider the set $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, y \geq 0 \right\}$ in $\mathbb{R}^2$. (i.e. the upper-half-plane)

<u>**0**-vector:</u> $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is in it!

<u>Closed under addition:</u> Suppose $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x' \\ y' \end{bmatrix}$ are in V, then $y \geq 0$ and $y' \geq 0$. Then $\mathbf{u} + \mathbf{v} = \begin{bmatrix} x + x' \\ y + y' \end{bmatrix}$. But since $y + y' \geq 0$, we get $\mathbf{u} + \mathbf{v}$ is in V

<u>Not closed under scalar multiplication:</u> For example, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in V, but $(-2)\mathbf{u} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ is not in V.

Note: I'm not going to be *too* harsh if you forgot to show that it's closed under addition, but I'll be very harsh if you didn't explicitly show to me that it's closed under scalar multiplication!!!

3. *(15 points)* Solve the following system of equations (or say it has no solutions):

$$\begin{cases} 2x + 2y + z = 2 \\ 3x + 4y + 2z = 3 \\ x + 2y - z = -3 \end{cases}$$

Write down the augmented matrix and row-reduce:

$$
\begin{aligned}
&\begin{bmatrix} 2 & 2 & 1 & 2 \\ 3 & 4 & 2 & 3 \\ 1 & 2 & -1 & -3 \end{bmatrix} \\
\rightarrow &\begin{bmatrix} 1 & 2 & -1 & -3 \\ 2 & 2 & 1 & 2 \\ 3 & 4 & 2 & 3 \end{bmatrix} \\
\rightarrow &\begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & -2 & 3 & 8 \\ 0 & -2 & 5 & 12 \end{bmatrix} \\
\rightarrow &\begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & -2 & 3 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \\
\rightarrow &\begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & -2 & 3 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix} \\
\rightarrow &\begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \\
\rightarrow &\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \\
\rightarrow &\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}
\end{aligned}
$$

Hence the solution is:

$$
\begin{cases} x = 1 \\ y = -1 \\ z = 2 \end{cases}
$$

4. *(20 points)* Solve the following system $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{bmatrix} 1 & 1 & 1 & -3 \\ 2 & 3 & 1 & -6 \\ -1 & 2 & -4 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 8 \\ 3 \end{bmatrix}$$

Write your answer in (parametric) vector form

$$\begin{bmatrix} 1 & 1 & 1 & -3 & 3 \\ 2 & 3 & 1 & -6 & 8 \\ -1 & 2 & -4 & 3 & 3 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 1 & 1 & -3 & 3 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 3 & -3 & 0 & 6 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 1 & 1 & -3 & 3 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 2 & -3 & 1 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now rewrite this as a system (**careful about the variables!**):

$$\begin{cases} x + 2z - 3t = 1 \\ y - z = 2 \\ (z = z) \\ (t = t) \end{cases}$$

$$\begin{cases} x = 1 - 2z + 3t \\ y = 2 + z \\ (z = z) \\ (t = t) \end{cases}$$

Hence, in vector form, this becomes:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 - 2z + 3t \\ 2 + z \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2z \\ z \\ z \\ 0 \end{bmatrix} + \begin{bmatrix} 3t \\ 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

5. *(15 points, 5 points each)*

(a) Calculate AB, or say that AB is undefined.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

This **is** defined, and AB is a 3×3 matrix:

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 3 \\ 2 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

(b) Calculate AB, or say that AB is undefined.

$$A = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 3 & 0 \end{bmatrix}$$

AB is **undefined** because A is 3×1 and B is 3×2, and $1 \neq 3$.

(c) Calculate A^2, where:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Note: If you're smart about this, you recognize A as the matrix which interchanges the 2 rows of a 2×2 matrix, so applying A

twice should just give you the identity matrix (i.e. the matrix that does 'nothing')!

6. *(15 points)* Find A^{-1} (or say 'A is not invertible') where:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

Form the (super) augmented matrix and row-reduce:

$$\begin{aligned}
\begin{bmatrix} A & I \end{bmatrix} &= \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & -4 & -7 & -2 & 0 & 1 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -4 & 1 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 7 & -2 \\ 0 & 0 & 1 & 2 & -4 & 1 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ -5 & 12 & -3 & -3 & 7 & -2 \\ 0 & 0 & 1 & 2 & -4 & 1 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & -3 & 7 & -2 \\ 0 & 0 & 1 & 2 & -4 & 1 \end{bmatrix} \\
&= \begin{bmatrix} I & A^{-1} \end{bmatrix}
\end{aligned}$$

Hence

$$A^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -3 & 7 & -2 \\ 2 & -4 & 1 \end{bmatrix}$$

7. *(15 points)* Find $det(A)$, where:

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 2 & 0 & 4 & 0 & 5 \\ 1 & 2 & 5 & -2 & 0 \\ 2 & 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

First expand along the second column (**be careful about the sign!**)

$$det(A) = -2 \begin{vmatrix} 1 & 0 & 3 & 1 \\ 2 & 4 & 0 & 5 \\ 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{vmatrix}$$

Then expand along the third column:

$$det(A) = (-2)(3) \begin{vmatrix} 2 & 4 & 5 \\ 2 & 3 & 1 \\ 0 & 1 & -1 \end{vmatrix} = (-6) \begin{vmatrix} 2 & 4 & 5 \\ 2 & 3 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$

Now expand along the last row:

$$det(A) = (-6) \left((-1) \begin{vmatrix} 2 & 5 \\ 2 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} \right) == (-6)(8+2) = -60$$

So $\boxed{det(A) = -60}$

8. *(10 points)* For the following matrix A, find a basis for $Nul(A)$, $Col(A)$, and find $Rank(A)$:

$$A = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 1 & 2 & -4 & 10 & 13 & -12 \\ 1 & -1 & -1 & 1 & 1 & -3 \\ 1 & -3 & 1 & -5 & -7 & 3 \\ 1 & -2 & 0 & 0 & -5 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\underline{Nul(A)}$ Since the right-hand-side is not in reduced row-echelon form, let's further row-reduce it:

$$\begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 4 & 5 & -6 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 0 & 9 & 2 \\ 0 & 1 & -1 & 0 & 7 & 3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(I first subtracted the second row from the first, and then subtracted 3 times the third row from the second and 4 times the third row from the first)

Now if $A\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ t \\ s \\ r \end{bmatrix}$, then we get:

$$\begin{cases} x - 2z + 9s + 2r = 0 \\ y - z + 7s + 3r = 0 \\ t - s - 2r = 0 \end{cases}$$

That is:

$$\begin{cases} x = 2z - 9s - 2r \\ y = z - 7s - 3r \\ t = s + 2r \end{cases}$$

Hence we get:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ t \\ s \\ r \end{bmatrix} = \begin{bmatrix} 2z - 9s - 2r \\ z - 7s - 3r \\ z \\ s + 2r \\ s \\ r \end{bmatrix} = z \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

And therefore:

$$Nul(A) = Span \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\underline{Col(A)}$ Notice that there are pivots in the first, second, and fourth columns, hence:

$$Col(A) = Span \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ 1 \\ -5 \\ 0 \end{bmatrix} \right\}$$

(Notice that you had to go back to the matrix A to find a basis for $Col(A)$)

$\underline{Rank(A)}$ There are 3 pivots, hence $Rank(A) = 3$.

9. *(5 points)* Define $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ (the space of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$) by:

$$T(y) = y'' - 5y' + 6y$$

Show that T is a linear transformation

$$T(y_1+y_2) = (y_1+y_2)'' - 5(y_1+y_2)' + 6(y_1+y_2) = y_1'' - 5y_1' + 6y_1 + y_2'' - 5y_2' + 6y_2 = T(y_1) + T(y_2)$$

$$T(cy) = (cy)'' - 5(cy)' + 6(cy) = c(y'' - 5y' + 6y) = cT(y)$$

10. *(5 points)*

(a) *(1 point)* If $T : V \to W$ is a one-to-one linear transformation and $T(\mathbf{x}) = \mathbf{0}$, what can you say about $\mathbf{x}$?

$\mathbf{x} = \mathbf{0}$

(b) *(4 points)* Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly *independent* vectors (in V) and $T : V \to W$ is a *one-to-one* linear transformation. Show that $T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{w})$ are also linearly independent.

Suppose $aT(\mathbf{u}) + bT(\mathbf{v}) + cT(\mathbf{w}) = \mathbf{0}$.

We want to show that $a = b = c = 0$

Then $T(a\mathbf{u} + b\mathbf{v} + c\mathbf{w}) = \mathbf{0}$ since T is linear

Hence $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$ since T is one-to-one (this is (a), with $\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$

Hence $a = b = c = 0$ since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent!

11. *(2 points)* Find $det(A)$, where:

$$A = \begin{bmatrix} 1 & x & x^2 & x^3 \\ 1 & y & y^2 & y^3 \\ 1 & z & z^2 & z^3 \\ 1 & t & t^2 & t^3 \end{bmatrix}$$

The trick is to **row-reduce** A (but you have to be **careful about the order!**

First, add (-1) times the first row to the second, third, and fourth rows while keeping the first row fixed (remember that this doesn't change the determinant):

$$det(A) = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & y-x & y^2-x^2 & y^3-x^3 \\ 0 & z-x & z^2-x^2 & z^3-x^3 \\ 0 & t-x & t^2-x^2 & t^3-x^3 \end{vmatrix}$$

Now notice that $y^2 - x^2 = (y-x)(y+x)$, and $y^3 - x^3 = (y-x)(y^2+xy+x^2)$, and so you can 'factor' out $(y-x)$ from the second row:

$$\begin{aligned} det(A) = & \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & (y-x) & (y-x)(y+x) & (y-x)(y^2+yx+x^2) \\ 0 & z-x & z^2-x^2 & z^3-x^3 \\ 0 & t-x & t^2-x^2 & t^3-x^3 \end{vmatrix} \\ = & (y-x)\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+yx+x^2 \\ 0 & z-x & z^2-x^2 & z^3-x^3 \\ 0 & t-x & t^2-x^2 & t^3-x^3 \end{vmatrix} \end{aligned}$$

But you can apply the exact same reasoning to the third and the fourth row, to get:

$$det(A) = (y-x)(z-x)(t-x)\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+xy+x^2 \\ 0 & 1 & z+x & z^2+xz+x^2 \\ 0 & 1 & t+x & t^2+xt+x^2 \end{vmatrix}$$

But now, add (-1) times the second row to the third row and the fourth row (all while leaving the second row fixed), to get:

$$det(A) = (y-x)(z-x)(t-x)\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+xy+x^2 \\ 0 & 0 & z-y & z^2-y^2+xz-xy \\ 0 & 0 & t-y & t^2-y^2+xt-xy \end{vmatrix}$$

But $z^2-y^2+xz-xy = (z-y)(z+y)+(z-y)x = (z-y)(z+y+x) = (z-y)(x+y+z)$, so you can factor out $(z-y)$ from the third row:

$$det(A) = (y-x)(z-x)(t-x)(z-y)\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+xy+x^2 \\ 0 & 0 & 1 & x+y+z \\ 0 & 0 & t-y & t^2-y^2+xt-xy \end{vmatrix}$$

Similarly, you can factor out $(t-y)$ from the fourth row:

$$det(A) = (y-x)(z-x)(t-x)(z-y)(t-y)\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+xy+x^2 \\ 0 & 0 & 1 & x+y+z \\ 0 & 0 & 1 & x+y+t \end{vmatrix}$$

Finally, add (-1) times the third row to the fourth row:

$$det(A) = (y-x)(z-x)(t-x)(z-y)(t-y)\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & y+x & y^2+xy+x^2 \\ 0 & 0 & 1 & x+y+z \\ 0 & 0 & 0 & t-z \end{vmatrix}$$

But this last matrix is upper-triangular, hence its determinant is $(1)(1)(1)(t-z)$, and we finally get:

$$\boxed{det(A) = (y-x)(z-x)(t-x)(z-y)(t-y)(t-z)}$$

The way to read this is as follows:

First fix x (the first variable), then take products of differences of the other variables with x, i.e. $(y-x)(z-x)(t-x)$.

Then fix y (the second variable), and take products of differences of all the other variables (<u>except for</u> x) with y, i.e. $(z-y)(t-y)$.

Finally, fix z (the next-to-last variable), and take products of differences of all the other variables (<u>except for</u> x and y) with z, i.e.

$(t - z)$.

And then take the product of everything you found to get:

$$det(A) = (y - x)(z - x)(t - x)(z - y)(t - y)(t - z)$$

In fact, there's a (natural) generalization of this! Google 'Vandermonde matrix' to learn more about this!