This is a closed book exam, no notes allowed. It consists of 6 problems, each worth 10 points, of which you must complete 5. **Choose one problem not to be graded by crossing it out in the box below.** If you forget to cross out a problem, we will roll a die to choose one for you.

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<th>Problem</th>
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1) Decide if the following statements are ALWAYS TRUE (T) or SOMETIMES FALSE (F). You do not need to justify your answers. (Correct answers receive 2 points, incorrect answers -2 points, blank answers 0 points.)

a) If $v_1, v_2, v_3, v_4$ are linearly independent vectors in $\mathbb{R}^6$, then $v_1 + v_2, v_3 - v_4$ are linearly independent vectors.

b) The following linear system is inconsistent

\[
\begin{align*}
-2x_1 &+ 4x_2 - 6x_3 &+ 8x_4 &= 10 \\
x_1 &- 2x_2 + 3x_3 &- 4x_4 &= -5
\end{align*}
\]

c) If $A$ is a $3 \times 2$ matrix and $B$ is a $2 \times 3$ matrix, then the rank of the $3 \times 3$ matrix $AB$ must be less than or equal to 2.

d) If two $m \times n$ matrices $A$ and $B$ have the same reduced row echelon form, then they have the same column spaces.

e) \[
\det \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 2 & 1 & 1 & 1 \\
1 & 1 & 3 & 1 & 1 \\
1 & 1 & 1 & 4 & 1 \\
1 & 1 & 1 & 1 & 5
\end{bmatrix} = 24
\]
2) Circle all of the answers that satisfy the questions below. It is possible that any number of
the answers (including none) satisfy the questions. (Complete solutions receive 2 points, partial
solutions 1 points, but any incorrect circled answer leads to 0 points.)

a) Let \( A \) be an \( m \times n \) matrix. Which of the following is equal to \( m \)?
   i) \( \text{rank}(A) \)
   ii) \( \text{dim} \text{Col}(A) + \text{dim} \text{Nul}(A) \)
   iii) \( \text{rank}(A^T) \)
   iv) \( \text{dim} \text{Col}(A^T) - \text{dim} \text{Nul}(A^T) \)
   v) \( \text{dim} \text{Col}(A^T) + \text{dim} \text{Nul}(A^T) \)

b) Which of the following matrices is in reduced row echelon form?
   i) \[
   \begin{bmatrix}
   1 & 1 & 2 \\
   0 & 0 & 1 \\
   \end{bmatrix}
   \]
   ii) \[
   \begin{bmatrix}
   1 & 2 & 0 \\
   0 & 0 & 1 \\
   \end{bmatrix}
   \]
   iii) \[
   \begin{bmatrix}
   1 & 0 & 2 & 1 \\
   0 & 0 & 1 \\
   0 & 0 & 0 \\
   \end{bmatrix}
   \]
   iv) \[
   \begin{bmatrix}
   1 & 0 & -2 & 1 \\
   0 & 1 & 0 & 0 \\
   0 & 0 & 1 & 1 \\
   \end{bmatrix}
   \]
   v) \[
   \begin{bmatrix}
   1 & -2 & 0 & 0 \\
   0 & 0 & 1 & 0 \\
   0 & 0 & 0 & 1 \\
   \end{bmatrix}
   \]

c) Which of the following conditions insures an \( m \times n \) matrix \( A \) is invertible?
   i) \( m = n \).
   ii) There exists an \( n \times m \) matrix \( B \) such that \( AB = I_m \).
   iii) The row echelon form of \( A \) has the same number of pivot rows as pivot columns.
   iv) \( Ax = b \) has a unique solution \( x \) for every \( b \).
   v) \( A \) is injective and surjective.

d) Which of the following \( T : \mathbb{R}^2 \to \mathbb{R} \) is a linear transformation?
   i) \( T(x, y) = x + y + 1 \)
   ii) \( T(x, y) = x - 2y \)
   iii) \( T(x, y) = x^2 + y^2 - (x + y)^2 \)
   iv) \( T(x, y) = 6(x + 1) + 2(y - 3) \)
   v) \( T(x, y) = 0 \)

e) Suppose \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) has 2-dimensional range and we know
   \[
   T(e_1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad T(e_3) = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}
   \]
   Which of the following is a possible value of \( T(e_2) \)?
   i) \[
   \begin{bmatrix}
   -1 \\
   -1 \\
   2 \\
   \end{bmatrix}
   \]
   ii) \[
   \begin{bmatrix}
   -2 \\
   1 \\
   1 \\
   \end{bmatrix}
   \]
   iii) \[
   \begin{bmatrix}
   3 \\
   -1 \\
   0 \\
   \end{bmatrix}
   \]
   iv) \[
   \begin{bmatrix}
   0 \\
   0 \\
   0 \\
   \end{bmatrix}
   \]
   v) \[
   \begin{bmatrix}
   2 \\
   0 \\
   -1 \\
   \end{bmatrix}
   \]
3) Consider the matrix

a) (5 points) Find bases for the column space and null space of

\[
A = \begin{bmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & -1 & -1 & 1
\end{bmatrix}
\]

b) (5 points) For what values of \( c \) is the vector

\[
v = \begin{bmatrix}
c \\
2c \\
c^2
\end{bmatrix}
\]

in the column space of \( A \)?
4) (10 points) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ satisfies the following:

\[
T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\]

Find the standard matrix of $T$. 

5) Decide if each of the following matrices is invertible, and either find its inverse or justify why it is not invertible.

a) (5 points)

\[
A = \begin{bmatrix}
1 & 0 & 1 \\
2 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

b) (5 points)

\[
B = \begin{bmatrix}
1 & 2 & -1 & 1 \\
-4 & 4 & 2 & 2 \\
-2 & -4 & -4 & -2 \\
1 & 2 & -2 & 1
\end{bmatrix}
\]
6) (10 points) Suppose that $v_1, \ldots, v_k$ are vectors in $\mathbb{R}^n$ and that $A$ is an $m \times n$ matrix. Prove that if $Av_1, \ldots, Av_k$ are linearly independent in $\mathbb{R}^m$, then $v_1, \ldots, v_k$ are linearly independent.