MATH 54 – FINAL EXAM

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Name: ______________________________________

Note: The reason this exam is so long is because it combines half of the second midterm, half of the third midterm, and half of the final exam that I gave during Summer 2012. This practice exam is not cumulative (even though the final is), so make sure to review the other exams as well!

1		10
2		5
3		10
4		10
5		25
6		10
7		10
8		10
9		35
10		5
11		10
12		5
13		5
Bonus		5
Total		150

Date: Friday, August 12, 2012.

1. *(10 points)* Define $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ (the space of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$) by:

$$T(y) = y'' - 5y' + 6y$$

(a) *(5 points)* Show that T is a linear transformation

(b) *(5 points)* Find a basis for $Ker(T)$ (or $Nul(T)$ if you wish).
Show that the basis you found is in fact a basis (i.e. is linearly independent and spans $Ker(T)$)!

Note: This is a tiny bit longer than you think!

2. *(5 points)* Find the largest *open* interval (a, b) on which the following differential equation has a unique solution:

$$(x-3)y'' + \left(\sqrt{x}\right) y' = \sqrt{x-1}$$

with

$$y(2) = 3, y'(2) = 1$$

3. *(10 points)* Solve the following differential equation:

$$y''' - 3y'' + 12y' - 10y = 0$$

4. *(10 points)* Solve the following differential equation:

$$y'' - 3y' + 2y = e^{3t}$$

5. *(25 points)* Solve the following system $\mathbf{x}' = A\mathbf{x}$, where:

$$A = \begin{bmatrix} 0 & 5 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Note: Show *all* your work!

(Continuation)

6. *(10 points)* Apply the Gram-Schmidt process to find an *orthonormal* basis of W, where:

$$W = Span \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

7. *(10 points)* Consider the space $C[-\pi, \pi]$ with the dot product:

$$f \cdot g = \int_{-\pi}^{\pi} f(t)g(t)dt$$

Find the orthogonal projection of $f(x) = x$ on

$$W = Span\{1, \cos(x), \cos(2x)\}$$

And use this to find a function g which is orthogonal to W.

Note: Don't waste *too* much time calculating the integrals, this should be quicker than you think!

8. *(10 points)* Consider the (inconsistent) system of equations $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}$$

(a) *(5 points)* Find the orthogonal projection of $\mathbf{b}$ on $Col(A)$

Hint: The columns of A are orthogonal!

(b) *(5 points)* Use your answer in (a) to find a least-squares solution to the system $A\mathbf{x} = \mathbf{b}$

9. *(35 points)* Find a solution to the following wave equation:

$$\begin{cases} u_{tt} = 9u_{xx} & 0 < x < \pi, \quad t > 0 \\ u_x(0,t) = u_x(\pi,t) = 0 & t > 0 \\ u(x,0) = x^2(\pi - x) & 0 < x < \pi \\ u_t(x,0) = 0 & 0 < x < \pi \end{cases}$$

Note: Make sure to show *all* your work, and make sure to do this problem from scratch.

(Scratch work)

10. *(5 points)* Consider $f(x) = x^2 + 1$ on $(-1, 1)$.

Draw the graph of $\mathcal{F}(x)$, the Fourier *sine* series of f on $(-4, 4)$.

11. *(10 points)* Consider $f(x) = \begin{cases} 0 & \text{on}(-1,0) \\ 1 & \text{on}(0,1) \end{cases}$.

Parseval's identity states that:

$$\sum_{m=0}^{\infty} (A_m)^2 + (B_m)^2 = \int_{-1}^{1} (f(x))^2$$

Where A_m and B_m are the (full) Fourier coefficients of f.

Calculate A_m and B_m and use this to calculate:

$$\sum_{m=1, m\ odd}^{\infty} \frac{1}{m^2} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} \cdots$$

12. *(5 points)* Use separation of variables to find the general solution to the following PDE:

$$\begin{cases} u_{xx} + u_{yy} = u \\ u(0,y) = u(1,y) = 0 \end{cases}$$

(where $u = u(x,y)$ and $0 < x < 1, 0 < y < 1$)

Hint: You can do this!!!

13. *(5 points)* Suppose $\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is orthonormal. Show that $\mathcal{B}$ is linearly independent!

Hint: Use hugging!

Note: Let me start the proof for you:

Suppose $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$.

Goal: Show that $a = b = c = 0$

Bonus *(5 points)* In this problem, we're going to use the Wronskian to find the general solution of a quite complicated differential equation! This should illustrate yet again the power of the Wronskian!

(a) Consider the differential equation:

$$y'' + P(t)y' + Q(t)y = 0$$

Recall the definition of the Wronskian (determinant):

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_2'(t)y_1(t) - y_1'(t)y_2(t)$$

Where y_1 and y_2 solve the above differential equation.

By differentiating $W(t)$ with respect to t, find a simple differential equation satisfied by $W(t)$ and solve it.

Note: You answer will involve the $\int$ sign!

Note: From now on, ignore the constants, i.e. in your answer in (a), set $C = 1$

(b) From (a), we get:

$$y_2'(t)y_1(t) - y_2(t)y_1'(t) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \text{(your answer from } (a)\text{)}$$

Solve for y_2 in terms of y_1.

Hint: Divide this equality by $(y_1(t))^2$ and recognize the left-hand-side as the derivative of a quotient, and hence solve for y_2 in terms of y_1. You answer will involve another $\int$ sign!

Note: Again, ignore the constants!

(c) Let's apply the result in (b) to the differential equation:

$$y'' - \tan(t)y' + 2y = 0$$

(here $P(t) = -\tan(t)$, $Q(t) = 2$)

One solution (by guessing) is given by $\boxed{y_1(t) = \sin(t)}$.

Use your answer in (b) to find *another* solution $y_2(t)$!

Note: Again, you can ignore the constants!

Hint: You should use the following facts, in order:

1) $\int \tan(t)dt = -\ln(\cos(t))$
2) At some point, multiply your integrand (the fct you're integrating) by $\frac{\cos(t)}{\cos(t)}$
3) The substitution $u = \frac{1}{\sin(t)}$
4) The formula $\frac{u^2}{1-u^2} = \frac{1}{1-u^2} - 1 = \frac{1}{2(1-u)} + \frac{1}{2(1+u)} - 1$
5) The function $\coth^{-1}(z) = \frac{1}{2}\ln\left|\frac{1-z}{1+z}\right|$
6) Try to have a formula *without* $\frac{1}{\sin(t)}$ (see how you can simplify the stuff inside the $\ln$)

(d) Notice that the equation $y'' - \tan(t)y' + 2y = 0$, although quite complicated, is still *linear*. What is the general solution of $y'' - \tan(t)y' + 2y = 0$? (no need to show your work here)