MATH 54 – FINAL EXAM – SOLUTIONS

PEYAM RYAN TABRIZIAN

1. *(10 points)* Define $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ (the space of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$) by:

$$T(y) = y'' - 5y' + 6y$$

   (a) *(5 points)* Show that T is a linear transformation

$$T(y_1+y_2) = (y_1+y_2)''-5(y_1+y_2)'+6(y_1+y_2) = y_1''-5y_1'+6y_1+y_2''-5y_2'+6y_2 = T(y_1)+T(y_2)$$

$$T(cy) = (cy)'' - 5(cy)' + 6(cy) = c(y'' - 5y' + 6y) = cT(y)$$

   (b) *(5 points)* Find a basis for $Ker(T)$ (or $Nul(T)$ if you wish). Show that the basis you found is in fact a basis (i.e. is linearly independent and spans $Ker(T)$)!

Remember $Ker(T) = \{y \mid T(y) = 0\} = \{y \mid y'' - 5y' + 6y = 0\}$

Now let's solve $y'' - 5y' + 6y = 0$. The auxiliary equation is $r^2 - 5r + 6 = (r-2)(r-3) = 0$, which gives $r = 2, r = 3$, hence the general solution is:

$$y(t) = Ae^{2t} + Be^{3t}$$

We claim that $\{e^{2t}, e^{3t}\}$ is a basis for $Ker(T)$. We already showed it spans $Ker(T)$, so all we need to show that it is linearly independent.

Date: Friday, August 12, 2012.

But the Wronskian matrix of e^{2t}, e^{3t} is $\tilde{W}(t) = \begin{bmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{bmatrix}$, hence $\tilde{W}(0) = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, so $W(0) = 3 - 2 = 1 \neq 0$, hence e^{2t} and e^{3t} are linearly independent!

Therefore $\{e^{2t}, e^{3t}\}$ is linearly independent and Spans $Ker(T)$, and hence it's a basis for $Ker(T)$.

2. *(5 points)* Find the largest *open* interval (a, b) on which the following differential equation has a unique solution:

$$(x-3)y'' + \left(\sqrt{x}\right) y' = \sqrt{x-1}$$

with

$$y(2) = 3, y'(2) = 1$$

First convert the equation in standard form:

$$y'' + \left(\frac{\sqrt{x}}{x-3}\right) y' = \frac{\sqrt{x-1}}{x-3}$$

Now let's look at the domain of each term:

The domain of $\frac{\sqrt{x}}{x-3}$ is $[0, 3) \cup (3, \infty)$ (i.e. all nonnegative real numbers except 3). The part of that interval which contains the initial condition 2 is $\boxed{[0, 3)}$

The domain of $\frac{\sqrt{x-1}}{x-3}$ is $[1, 3) \cup (3, \infty)$. The part of that which contains the initial condition 2 is $\boxed{[1, 3)}$

And if you intersect the two domains you found you get $[1, 3)$, and hence the answer is $\boxed{(1, 3)}$ (remember we want an open interval!).

3. *(10 points)* Solve the following differential equation:

$$y''' - 3y'' + 12y' - 10y = 0$$

The auxiliary equation is $r^3 - 3r^2 + 12r - 10 = 0$.

Now, by the rational roots theorem, we know that if the above polynomial has a rational root, then $r = \frac{a}{b}$, where a divides the constant term -42 and b divides the leading term 1.

The only integers which divide -10 are $\pm 1, \pm 2, \pm 5, \pm 10$
And the only integers which divide 1 are ± 1. Hence our guesses are: $\pm 1, \pm 2, \pm 5, \pm 10$.

If you plug-and-chug, you eventuall figure out that $r = 1$ works (the first guess!), i.e. $r = 1$ is a root of the auxiliary polynomial.

Now all you have to do is use long division and divide $r^3 - 3r^2 + 12r - 10$ by $r - 1$

$$\begin{array}{rrrrr}
 & X^2 & -2X & +10 & \\
\cline{2-5}
X-1 \big) & X^3 - 3X^2 & +12X & -10 & \\
 & -X^3 + X^2 & & & \\
\cline{2-2}
 & -2X^2 & +12X & & \\
 & 2X^2 & -2X & & \\
\cline{2-3}
 & & 10X & -10 & \\
 & & -10X & +10 & \\
\cline{3-4}
 & & & 0 &
\end{array}$$

In other words, $r^3 - 3r^2 + 12r - 10 = (r-1)(r^2 - 2r + 10)$

Now $r^2 - 2r + 10 = 0 \Rightarrow r = \frac{2 \pm \sqrt{-36}}{2} = 1 \pm 3i$.
Hence, the roots of $r^3 - 3r^2 + 12r - 10$ are $r = 1$ and $r = 1 \pm 3i$. Hence the general solution of the differential equation is:

$$y(t) = Ae^t + Be^t \cos(3t) + Ce^t \sin(3t)$$

4. *(10 points)* Solve the following differential equation:

$$y'' - 3y' + 2y = e^{3t}$$

First let's solve the homogeneous equation: $r^2 - 3r + 2 = (r - 1)(r - 2) = 0 \Rightarrow r = 1, r = 2$, hence $y_0(t) = Ae^t + Be^{2t}$

Now for the particular solution y_p, let's guess $y_p(t) = Ce^{3t}$. If you plug this into the original equation, you get:

$$9Ce^{3t} - 9Ce^{3t} + 2Ce^{3t} = e^{3t} \Rightarrow 2C = 1 \Rightarrow C = \frac{1}{2}$$

Hence $y_p(t) = \frac{1}{2}e^{3t}$, and $\boxed{y(t) = Ae^t + Be^{2t} + \frac{1}{2}e^{3t}}$

5. *(20 points)* Solve the following system $\mathbf{x}' = A\mathbf{x}$, where:

$$A = \begin{bmatrix} 0 & 5 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

<u>Eigenvalues:</u>

$$det(\lambda I - A)) = \begin{vmatrix} \lambda & -5 & 0 \\ 1 & \lambda - 4 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)\left[\lambda(\lambda - 4) + 5\right] = (\lambda - 2)(\lambda^2 - 4\lambda + 5) = 0$$

Which gives $\lambda = 2$ and $\lambda^2 - 4\lambda + 5 = 0$, so $(\lambda - 2)^2 + 1 = 0$, so $\lambda = 2 \pm i$

Hence the eigenvalues are $\boxed{\lambda = 2, 2 \pm i}$

<u>Eigenvectors:</u>

<u>$\lambda = 2$</u>

$$Nul(2I - A) = Nul \begin{bmatrix} -2 & 5 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = Nul \begin{bmatrix} 1 & 2 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix} = Nul \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

But then $x = 0, y = 0$, and so:
And so:

$$Nul(2I - A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \right\} = Span \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

This tells us that $\mathbf{x}(t) = e^{2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a solution to the differential equation!

<u>$\lambda = 2 + i$</u>

$$Nul((2 + i)I - A) = Nul \begin{bmatrix} 2 + i & -5 & 0 \\ 1 & -2 + i & 0 \\ 0 & 0 & i \end{bmatrix} = Nul \begin{bmatrix} 1 & \frac{-5}{2+i} & 0 \\ 1 & -2 + i & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

However, $\frac{-5}{2+i} = \frac{-5(2-i)}{(2+i)(2-i)} = \frac{-5(2-i)}{4+1} = -2+i$, so:

$$Nul((2+i)I - A) = Nul \begin{bmatrix} 1 & -2+i & 0 \\ 1 & -2+i & 0 \\ 0 & 0 & 1 \end{bmatrix} = Nul \begin{bmatrix} 1 & -2+i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

But then $x + (-2+i)y = 0$ and $z = 0$, so $x = (2-i)y$, and:

$$Nul((2+i)I - A) = \left\{ \begin{bmatrix} (2-i)y \\ y \\ 0 \end{bmatrix} \right\} = Span \left\{ \begin{bmatrix} 2-i \\ 1 \\ 0 \end{bmatrix} \right\}$$

This tells us that $\mathbf{x}(t) = e^{(2+i)t} \begin{bmatrix} 2-i \\ 1 \\ 0 \end{bmatrix}$ is a solution to the differential equation! But we can simplify this:

$$\begin{aligned} e^{(2+i)t} \begin{bmatrix} 2-i \\ 1 \\ 0 \end{bmatrix} &= \left(e^{2t}\cos(t) + ie^{2t}\sin(t)\right) \left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right) \\ &= \left(e^{2t}\cos(t) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - e^{2t}\sin(t) \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right) + i \left(e^{2t}\sin(t) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + e^{2t}\cos(t) \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right) \end{aligned}$$

And splitting into real and imaginary parts, and using the solution found for $\lambda = 2$, we get that:

General Solution:

$$\mathbf{x}(t) = Ae^{2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + B \left(e^{2t}\cos(t) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - e^{2t}\sin(t) \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right) + C \left(e^{2t}\sin(t) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + e^{2t}\cos(t) \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right)$$

6. *(10 points)* Apply the Gram-Schmidt process to find an *orthonormal* basis of W, where:

$$W = Span \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Step 1: Let $\mathbf{v_1} = \mathbf{u_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Step 2: Calculate:

$$\hat{\mathbf{u_2}} = \left(\frac{\mathbf{u_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right) \mathbf{v_1} = \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

And let:

$$\mathbf{v_2} = \mathbf{u_2} - \hat{\mathbf{u_2}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \sim \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Step 3: Calculate:

$$\hat{\mathbf{u_3}} = \left(\frac{\mathbf{u_3} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right) \mathbf{v_1} + \left(\frac{\mathbf{u_3} \cdot \mathbf{v_2}}{\mathbf{v_2} \cdot \mathbf{v_2}}\right) \mathbf{v_2} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

And let:

$$\mathbf{v_3} = \mathbf{u_3} - \hat{\mathbf{u_3}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Step 4: Normalize:

$$\mathbf{w_1} = \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w_2} = \frac{\mathbf{v_2}}{\|\mathbf{v_2}\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \mathbf{w_3} = \frac{\mathbf{v_3}}{\|\mathbf{v_3}\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Answer:

$$\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \right\}$$

7. *(10 points)* Consider the space $C[-\pi, \pi]$ with the dot product:

$$f \cdot g = \int_{-\pi}^{\pi} f(t)g(t)dt$$

Find the orthogonal projection of $f(x) = x$ on

$$W = Span\{1, \cos(x), \cos(2x)\}$$

And use this to find a function g which is orthogonal to W.

Note: Don't waste *too* much time calculating the integrals, this should be quicker than you think!

$$\begin{aligned}
\hat{f} &= \left(\frac{x \cdot 1}{1 \cdot 1}\right) 1 + \left(\frac{x \cdot \cos(x)}{\cos(x) \cdot \cos(x)}\right) \cos(x) + \left(\frac{x \cdot \cos(2x)}{\cos(2x) \cdot \cos(2x)}\right) \cos(2x) \\
&= \left(\frac{\int_{-\pi}^{\pi} x}{\int_{-\pi}^{\pi} 1}\right) 1 + \left(\frac{\int_{-\pi}^{\pi} x\cos(x)}{\int_{-\pi}^{\pi} \cos^2(x)}\right) \cos(x) + \left(\frac{\int_{-\pi}^{\pi} x\cos(2x)}{\int_{-\pi}^{\pi} \cos^2(2x)}\right) \cos(2x) \\
&= 0
\end{aligned}$$

Here all the integrals on the numerator are 0 because the integral of an odd function over $[-\pi, \pi]$ is 0.

And finally:

$$g(x) = f(x) - \hat{f}(x) = x$$

8. *(10 points)* Consider the (inconsistent) system of equations $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}$$

(a) *(5 points)* Find the orthogonal projection of $\mathbf{b}$ on $Col(A)$

Hint: The columns of A are orthogonal!

Let $\mathbf{a_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{a_2} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ be the columns of A. Then:

$$\begin{aligned} \hat{\mathbf{b}} &= \left(\frac{\mathbf{b} \cdot \mathbf{a_1}}{\mathbf{a_1} \cdot \mathbf{a_1}}\right) \mathbf{a_1} + \left(\frac{\mathbf{b} \cdot \mathbf{a_2}}{\mathbf{a_2} \cdot \mathbf{a_2}}\right) \mathbf{a_2} \\ &= \left(\frac{8}{4}\right) \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \left(\frac{12}{4}\right) \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \\ &= \mathbf{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \mathbf{3} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ -1 \\ 1 \\ 5 \end{bmatrix} \end{aligned}$$

Note: You *couldn't* use $\hat{\mathbf{b}} = AA^T\mathbf{b}$ because A is not orthogonal (its columns are not orthonormal). However, once you normalize the columns of A to get A', you could also use $\hat{\mathbf{b}} = A'(A')^T$

(b) *(5 points)* Use your answer in (a) to find a least-squares solution to the system $A\mathbf{x} = \mathbf{b}$

We need to find $\widetilde{\mathbf{x}}$ such that $A\widetilde{\mathbf{x}} = \hat{\mathbf{b}}$, where $\hat{\mathbf{b}}$ is as in (a), so:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}, \widetilde{\mathbf{x}} = \begin{bmatrix} \widetilde{x} \\ \widetilde{y} \end{bmatrix}, \hat{\mathbf{b}} = \begin{bmatrix} 5 \\ -1 \\ 1 \\ 5 \end{bmatrix}$$

Now row-reduce:

$$\begin{bmatrix} 1 & 1 & 5 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Which gives $\boxed{\widetilde{\mathbf{x}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}}$

Note: Another way to do this is to notice that the coefficients of the linear combination in (a) are $\mathbf{2}$ and $\mathbf{3}$. But that corresponds precisely to $\mathbf{x}$ (i.e. $\mathbf{x}$ is the vector of coefficients we need to apply to the columns of A to produce $\mathbf{b}$), hence $\boxed{\widetilde{\mathbf{x}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}}$

9. *(35 points)* Find a solution to the following wave equation:

(1)
$$\begin{cases} u_{tt} = & 9u_{xx} & 0 < x < \pi, \quad t > 0 \\ u_x(0,t) = & u_x(\pi,t) = 0 & t > 0 \\ u(x,0) = & x^2(\pi - x) & 0 < x < \pi \\ u_t(x,0) = & 0 & 0 < x < \pi \end{cases}$$

Note: Make sure to show *all* your work, and make sure to do this problem from scratch.

Step 1: Separation of variables. Suppose:

(2)
$$u(x,t) = X(x)T(t)$$

Plug (2) into the differential equation (1), and you get:

$$\begin{aligned} (X(x)T(t))_{tt} =& 9\,(X(x)T(t))_{tt} \\ X(x)T''(t) =& 9X''(x)T(t) \end{aligned}$$

Rearrange and get:

(3)
$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{9T(t)}$$

Now $\frac{X''(x)}{X(x)}$ *only* depends on x, but by (3) *only* depends on t, hence it is constant:

(4)
$$\begin{aligned} \frac{X''(x)}{X(x)} =& \lambda \\ X''(x) =& \lambda X(x) \end{aligned}$$

Also, we get:

(5)
$$\begin{aligned}\frac{T''(t)}{9T(t)} &= \lambda \\ T''(t) &= 9\lambda T(t)\end{aligned}$$

but we'll only deal with that later (Step 4)

Step 2: Consider (4):

$$X''(x) = \lambda X(x)$$

Now use the **boundary conditions** in (1):

$$u_x(0,t) = X'(0)T(t) = 0 \Rightarrow X'(0)T(t) = 0 \Rightarrow X'(0) = 0$$

$$u_x(\pi,t) = X'(\pi)T(t) = 0 \Rightarrow X'(\pi)T(t) = 0 \Rightarrow X'(\pi) = 0$$

Hence we get:

(6)
$$\begin{cases} X''(x) = \lambda X(x) \\ X'(0) = 0 \\ X'(\pi) = 0 \end{cases}$$

Step 3: Eigenvalues/Eigenfunctions. The auxiliary polynomial of (6) is $p(\lambda) = r^2 - \lambda$

Now we need to consider 3 cases:

<u>Case 1:</u> $\lambda > 0$, then $\lambda = \omega^2$, where $\omega > 0$

Then:

$$r^2 - \lambda = 0 \Rightarrow r^2 - \omega^2 = 0 \Rightarrow r = \pm\omega$$

Therefore:

$$X(x) = Ae^{\omega x} + Be^{-\omega x}$$

And

$$X'(x) = A\omega e^{\omega x} - B\omega e^{-\omega x}$$

Now use $X'(0) = 0$ and $X'(\pi) = 0$:

$$X'(0) = A\omega - B\omega = 0 \Rightarrow B\omega = A\omega \Rightarrow A = B \Rightarrow X(x) = Ae^{\omega x} + Ae^{-\omega x}$$

$$X'(\pi) = 0 \Rightarrow A\omega e^{\omega\pi} - Ae^{-\omega\pi} = 0 \Rightarrow Ae^{\omega\pi} = Ae^{-\omega\pi} \Rightarrow e^{\omega\pi} = e^{-\omega\pi} \Rightarrow \omega\pi = -\omega\pi \Rightarrow \omega = 0$$

But this is a **contradiction**, as we want $\omega > 0$.

Case 2: $\lambda = 0$, then $r = 0$, and:

$$X(x) = Ae^{0x} + Bxe^{0x} = A + Bx$$

And:

$$X'(x) = B$$

So:

$$X'(0) = 0 \Rightarrow B = 0 \Rightarrow X(x) = A$$

$$X'(\pi) = 0 \Rightarrow 0 = 0$$

Which is perfectly valid (not a contradiction), so $\boxed{\lambda = 0}$ works and $\boxed{X(x) = A}$

Case 3: $\lambda < 0$, then $\lambda = -\omega^2$, and:

$$r^2 - \lambda = 0 \Rightarrow r^2 + \omega^2 = 0 \Rightarrow r = \pm\omega i$$

Which gives:

$$X(x) = A\cos(\omega x) + B\sin(\omega x)$$

So:

$$X'(x) = -A\omega\sin(\omega x) + B\omega\cos(\omega x)$$

Again, using $X'(0) = 0$, $X'(\pi) = 0$, we get:

$$X'(0) = B = 0 \Rightarrow \boxed{X(x) = A\cos(\omega x)}, \text{and} X'(x) = -A\omega\sin(\omega x)$$

$$X'(\pi) = -A\omega\sin(\omega\pi) = 0 \Rightarrow \sin(\omega\pi) = 0 \Rightarrow \omega = m, \quad (m = 1, 2, \cdots)$$

This tells us that (combined with Case 2):

(7)
$$\text{Eigenvalues:} \lambda = -\omega^2 = -m^2 \quad (m = 0, 1, 2, \cdots)$$
$$\text{Eigenfunctions:} X(x) = \cos(\omega x) = \cos(mx)$$

Step 4: Deal with (5), and remember that $\lambda = -m^2$:

$$T''(t) = 9\lambda T(t)$$

Case 1: $\lambda = 0$

Aux: $r^2 = 0 \Rightarrow r = 0$ (double root), which gives:

$$T_0(t) = \widetilde{A_0} + \widetilde{B_0} t$$

Case 2: $\lambda = -m^2$ $(m = 1, \cdots)$

Aux: $r^2 = -9m^2 \Rightarrow r = \pm 3mi \qquad (m = 1, 2, \cdots)$

$$T_m(t) = \widetilde{A_m} \cos(3mt) + \widetilde{B_m} \sin(3mt)$$

Step 5: Take linear combinations:

(8)
$$u(x,t) = \sum_{m=0}^{\infty} T_m(t) X_m(x) = \left(\widetilde{A_0} + \widetilde{B_0} t\right) \cos(0x) + \sum_{m=1}^{\infty} \left(\widetilde{A_m} \cos(3mt) + \widetilde{B_m} \sin(3mt)\right) \cos(mx)$$

Step 6: Use the initial condition $u(x,0) = x^2(\pi - x)$ in (1):

Plug in $t = 0$ in (8), and you get:

(9)
$$u(x,0) = \sum_{m=0}^{\infty} \widetilde{A_m} \cos(mx) = x^2(\pi - x) \qquad \text{on}(0,\pi)$$

Hence we need to find a Fourier cosine series, with $f(x) = x^2(\pi - x) = \pi x^2 - x^3$, so 'evenify' f to get $\widetilde{f}$, and:

$$
\begin{aligned}
\widetilde{A_0} =& \frac{\int_{-\pi}^{\pi} \widetilde{f}(x)}{\int_{-\pi}^{\pi} 1} \\
=& \frac{2\int_0^{\pi} \pi x^2 - x^3}{2\pi} \\
=& \frac{1}{\pi}\left[\pi \frac{x^3}{3} - \frac{x^4}{4}\right]_0^{\pi} \\
=& \frac{1}{\pi}\left(\frac{\pi^4}{3} - \frac{\pi^4}{4} - 0 + 0\right) \\
=& \frac{\pi^3}{12}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{A_m} =& \frac{\int_{-\pi}^{\pi} \widetilde{f}(x)\cos(mx)}{\int_{-\pi}^{\pi} \cos^2(mx)} \\
=& \frac{2\int_0^{\pi} (\pi x^2 - x^3)\cos(mx)}{\pi} \\
=& \frac{2}{\pi}\left[\left(\pi x^2 - x^3\right)\frac{\sin(mx)}{m} - (2\pi x - 3x^2)\frac{-\cos(mx)}{m^2} + (2\pi - 6x)\frac{-\sin(mx)}{m^3} - (-6)\frac{\cos(mx)}{m^4}\right]_0^{\pi} \\
=& \frac{2}{\pi}\left(0 + (2\pi^3 - 3\pi^3)\frac{\cos(\pi m)}{m^2} - 0 - 0 + 6\frac{\cos(\pi m) - 1}{m^4}\right) \\
=& \frac{2}{\pi}\left(\frac{-\pi^3(-1)^m}{m^2} + \frac{6((-1)^m - 1)}{m^4}\right) \\
=& \frac{-2\pi^2(-1)^m}{m^2} + \frac{12((-1)^m - 1)}{\pi(m)^4}
\end{aligned}
$$

(for this, we used tabular integration, as well as the fact that the $\sin$ terms are 0)

Step 7: Use the initial condition: $\frac{\partial u}{\partial t}(x,0) = 2\cos(2x) + 8\cos(4x)$ in (1)

First differentiate (8) with respect to t:

(10)

$$
\frac{\partial u}{\partial t}(x,t) = \widetilde{B_0} + \sum_{m=1}^{\infty}\left(-3m\widetilde{A_m}\sin(mt) + 3m\widetilde{B_m}\cos(mt)\right)\cos(mx)
$$

Now plug in $t = 0$ in (10):

$$\frac{\partial u}{\partial t}(x,0) = \widetilde{B_0} + \sum_{m=1}^{\infty} 3m\widetilde{B_m}\cos(mx) = 0 \tag{11}$$

By linear independence, all the coefficients are equal to 0, and hence you get: $\widetilde{B_m} = 0$

Step 8: Conclude using (8) and the coefficients $\widetilde{A_m}$ and $\widetilde{B_m}$ you found:

$$u(x,t) = \widetilde{A_0} + \widetilde{B_0}t + \sum_{m=1}^{\infty}\left(\widetilde{A_m}\cos(3mt) + \widetilde{B_m}\sin(3mt)\right)\cos(mx) \tag{12}$$

where:

$$\widetilde{A_0} = \frac{\pi^3}{12}$$

$$\widetilde{A_m} = \frac{-2\pi^2(-1)^m}{m^2} + \frac{12((-1)^m - 1)}{\pi m^4}, \quad m = 1, 2, \cdots$$

and

$$\widetilde{B_m} = 0 \quad m = 0, 1, 2, \cdots$$

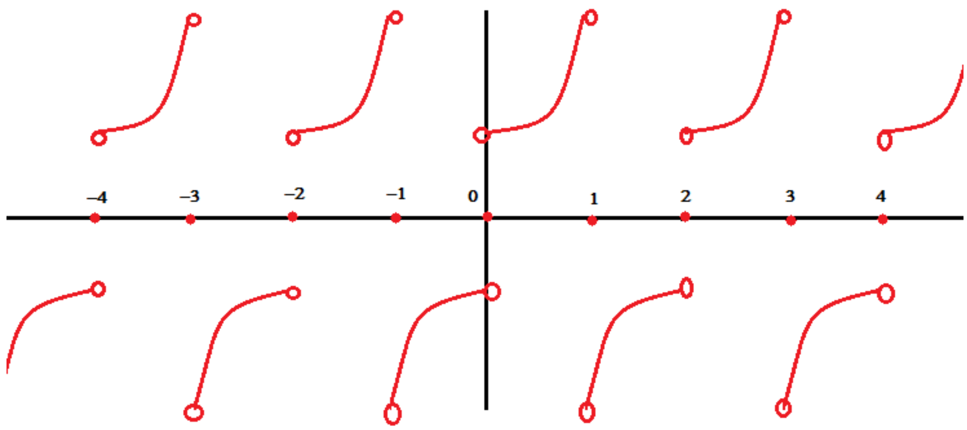
That is:

$$u(x,t) = \frac{\pi^3}{12} + \sum_{m=1}^{\infty}\left(\frac{-2\pi^2(-1)^m}{m^2} + \frac{12((-1)^m - 1)}{\pi m^4}\right)\cos(3mt)\cos(mx)$$

10. *(5 points)* Consider $f(x) = x^2 + 1$ on $(-1, 1)$.

Draw the graph of $\mathcal{F}(x)$, the Fourier *sine* series of f on $(-4, 4)$.

For this, just 'oddify' f and repeat the graph of f:
54/Math 54 Summer/Exams/Finalgraph.png



11. *(10 points)* Consider $f(x) = \begin{cases} 0 & \text{on}(-1,0) \\ 1 & \text{on}(0,1) \end{cases}$.

Parseval's identity states that:

$$\sum_{m=0}^{\infty} (A_m)^2 + (B_m)^2 = \int_{-1}^{1} (f(x))^2$$

Where A_m and B_m are the (full) Fourier coefficients of f.

Calculate A_m and B_m and use this to calculate:

$$\sum_{m=1, m\ odd}^{\infty} \frac{1}{m^2} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} \cdots$$

$$A_0 = \frac{\int_{-1}^{1} f(x)}{\int_{-1}^{1} 1} = \frac{\int_0^1 1}{2} = \frac{1}{2}$$

$$\begin{aligned} A_m =& \frac{\int_{-1}^{1} f(x)\cos(\pi m x)}{\int_{-1}^{1} \cos^2(\pi m x)} \\ =& \frac{\int_0^1 \cos(\pi m x)}{1} \\ =& \left[\frac{\sin(\pi m x)}{\pi m}\right]_0^1 \\ =& 0 \end{aligned}$$

(We used the fact that $f \equiv 0$ on $(-1,0)$)

$$B_0 = 0$$

$$\begin{aligned}
B_m =& \frac{\int_{-1}^{1} f(x)\sin(\pi m x)}{\int_{-1}^{1}\sin^2(\pi m x)} \\
=& \frac{\int_0^1 \cos(\pi m x)}{1} \\
=& \left[\frac{-\cos(\pi m x)}{\pi m}\right]_0^1 \\
=& \frac{-1}{\pi m}(\cos(\pi m) - 1) \\
=& \frac{-1}{\pi m}((-1)^m - 1)
\end{aligned}$$

(We used the fact that $f \equiv 0$ on $(-1,0)$)

Now, using Parseval's identity, we get:

$$\begin{aligned}
\sum_{m=0}^{\infty} A_m^2 + B_m^2 =& \int_{-1}^{1}(f(x))^2 \\
A_0^2 + B_0^2 + \sum_{m=1}^{\infty} A_m^2 + B_m^2 =& \int_0^1 1 \\
\left(\frac{1}{2}\right)^2 + 0^2 + \sum_{m=1}^{\infty} 0^2 + \left(\frac{-1}{\pi m}((-1)^m - 1)\right) =& 1 \\
\sum_{m=1}^{\infty} \frac{1}{\pi^2 m^2}((-1)^m - 1)^2 =& 1 - \frac{1}{4} = \frac{3}{4} \\
\sum_{m=1}^{\infty} \frac{((-1)^m - 1)^2}{m^2} =& \frac{3\pi^2}{4}
\end{aligned}$$

And finally, to conclude, notice that $(-1)^m - 1 = 0$ if m is even and $= 2$ if m is odd, hence:

$$\sum_{m=1, m\ odd}^{\infty} \frac{2^2}{m^2} = \frac{3\pi^2}{4}$$

$$\sum_{m=1, m\ odd}^{\infty} \frac{1}{m^2} = \frac{3\pi^2}{4(4)} = \frac{3\pi^2}{16}$$

12. *(5 points)* Use separation of variables to find the general solution to the following PDE:

$$\begin{cases} u_{xx} + u_{yy} = u \\ u(0,y) = u(1,y) = 0 \end{cases}$$

(where $u = u(x,y)$ and $0 < x < 1, 0 < y < 1$)

Hint: You can do this!!!

Suppose $u(x,y) = X(x)Y(y)$. Then plug this into the above equation:

$$(X(x)Y(y))_{xx} + (X(x)Y(y))_{yy} = X(x)Y(y)$$

$$X''(x)Y(y) + X(x)Y''(y) = X(x)Y(y)$$

And divide all the sides by $X(x)Y(y)$:

$$\frac{X''(x)\cancel{Y(y)}}{X(x)\cancel{Y(y)}} + \frac{\cancel{X(x)}Y''(y)}{\cancel{X(x)}Y(y)} = 1$$

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 1$$

$$\frac{X''(x)}{X(x)} = 1 - \frac{Y''(y)}{Y(y)} = \lambda$$

Hence: $\boxed{X''(x) = \lambda X(x)}$ (and $Y''(y) = (1-\lambda)Y(y)$):

And as usual, we get that $X(0) = 0$ and $X(1) = 0$, and if we do the $3-$cases business as usual, we find that: $\boxed{\lambda = -(\pi m)^2 \text{ and } X(x) = \sin(\pi m x)}$ $(m = 1, 2, \cdots)$

Now go back to $Y''(y) = (1-\lambda)Y(y) = (1 + (\pi m)^2)Y(y)$. The auxiliary equation is $r^2 = 1 + (\pi m)^2$, which gives $r = \pm\sqrt{1 + (\pi m)^2}$, and hence:

$$Y(y) = \widetilde{A_m} e^{\sqrt{1+(\pi m)^2} y} + \widetilde{B_m} e^{-\sqrt{1+(\pi m)^2} y}$$

And hence:

$$X(x)Y(y) = \left(\widetilde{A_m} e^{\sqrt{1+(\pi m)^2} y} + \widetilde{B_m} e^{-\sqrt{1+(\pi m)^2} y}\right) \sin(\pi m x)$$

And finally take linear combinations:

$$u(x,y) = \sum_{m=1}^{\infty} \left(\widetilde{A_m} e^{\sqrt{1+(\pi m)^2} y} + \widetilde{B_m} e^{-\sqrt{1+(\pi m)^2} y}\right) \sin(\pi m x)$$

13. *(5 points)* Suppose $\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is orthonormal. Show that $\mathcal{B}$ is linearly independent!

Hint: Use hugging!

Note: Let me start the proof for you:

Suppose $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$.

Goal: Show that $a = b = c = 0$

First dot the above equation with $\mathbf{u}$ and use orthonormality:

$$\begin{aligned}(a\mathbf{u} + b\mathbf{v} + c\mathbf{w}) \cdot \mathbf{u} =& \mathbf{0} \cdot \mathbf{u} = 0 \\ a\mathbf{u} \cdot \mathbf{u} + b\mathbf{v} \cdot \mathbf{u} + c\mathbf{w} \cdot \mathbf{u} =& 0 \\ a(1) + b(0) + c(0) =& 0 \\ a =& 0\end{aligned}$$

Hence $b\mathbf{v} + c\mathbf{w} = \mathbf{0}$. Now dot this with $\mathbf{v}$ and use orthonormality:

$$\begin{aligned}(b\mathbf{v} + c\mathbf{w}) \cdot \mathbf{v} =& \mathbf{0} \cdot \mathbf{v} = 0 \\ b\mathbf{v} \cdot \mathbf{v} + c\mathbf{w} \cdot \mathbf{v} =& 0 \\ b(1) + c(0) =& 0 \\ b =& 0\end{aligned}$$

Hence $c\mathbf{w} = \mathbf{0}$. Finally, dot this with $\mathbf{w}$:

$$\begin{aligned}c\mathbf{w} \cdot \mathbf{w} =& \mathbf{0} \cdot \mathbf{w} = 0 \\ c(1) =& 0 \\ c =& 0\end{aligned}$$

Hence $a = b = c = 0$, and we're done!

Bonus *(5 points)*

(a) Consider the differential equation:

$$y'' + P(t)y' + Q(t)y = 0$$

Recall the definition of the Wronskian determinant:

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_2'(t)y_1(t) - y_1'(t)y_2(t)$$

Where y_1 and y_2 solve the above differential equation.

By differentiating $W(t)$ with respect to t, find a simple differential equation satisfied by $W(t)$ and solve it. You answer will involve the $\int$ sign!

$$\begin{aligned} W'(t) =& y_2''(t)y_1(t) + \cancel{y_2'(t)y_1'(t)} - y_1''(t)y_2(t) - \cancel{y_1'(t)y_2'(t)} \\ =& y_2''(t)y_1(t) - y_1''(t)y_2(t) \\ =& \left(-P(t)y_2'(t) - Q(t)y_2(t)\right) y_1(t) + \left(P(t)y_1'(t) + Q(t)y_1(t)\right) y_2(t) \\ =& - P(t)y_2'(t)y_1(t) - \cancel{Q(t)y_1(t)y_2(t)} + P(t)y_1'(t)y_2(t) + \cancel{Q(t)y_1(t)y_2(t)} \\ =& - P(t)\left(y_2'(t)y_1(t) - y_1'(t)y_2(t)\right) \\ =& - P(t)W(t) \end{aligned}$$

Hence $W'(t) = -P(t)W(t)$, so $\boxed{W(t) = Ce^{-\int P(t)dt}}$

(b) Hence, by (a), we know:

$$y_2'(t)y_1(t) - y_2(t)y_1'(t) = \text{your answer in } (a)$$

Divide this equality by $(y_1(t))^2$ and recognize the left-hand-side as the derivative of a quotient, and hence solve for y_2 in terms of y_1. You answer will involve another $\int$ sign!

We have:

$$\begin{aligned}
y_2'(t)y_1(t) - y_2(t)y_1'(t) &= e^{-\int P(t)dt} \\
\frac{y_2'(t)y_1(t) - y_2(t)y_1'(t)}{\left(y_1(t)\right)^2} &= \frac{e^{-\int P(t)dt}}{\left(y_1(t)\right)^2} \\
\left(\frac{y_2(t)}{y_1(t)}\right)' &= \frac{e^{-\int P(t)dt}}{\left(y_1(t)\right)^2} \\
\frac{y_2(t)}{y_1(t)} &= \int \frac{e^{-\int P(t)dt}}{\left(y_1(t)\right)^2} \\
y_2(t) &= \left(\int \frac{e^{-\int P(t)dt}}{\left(y_1(t)\right)^2}\right) y_1(t)
\end{aligned}$$

(c) Let's apply the result in (b) to the differential equation:

$$y'' - \tan(t)y' + 2y = 0$$

(here $P(t) = -\tan(t)$, $Q(t) = 2$)

One solution (by guessing) is given by $y_1(t) = \sin(t)$. Use your answer to (b) to find *another* solution $y_2(t)$!

Hint: You may use the following facts: $\int \tan(t)dt = -\ln\left(\cos(t)\right)$, the substitution $u = \frac{1}{\sin(t)}$, and finally the formula $\frac{u^2}{1-u^2} = \frac{1}{1-u^2} - 1 = \frac{1}{2(1-u)} + \frac{1}{2(1+u)} - 1$.

We have:

$$
\begin{aligned}
y_2(t) &= \left(\int \frac{e^{-\int P(t)dt}}{(y_1(t))^2} \right) y_1(t) \\
&= \left(\int \frac{e^{\int \tan(t)dt}}{\sin^2(t)} \right) \sin(t) \\
&= \left(\int \frac{e^{-\ln(\cos(t))}}{\sin^2(t)} \right) \sin(t) \\
&= \left(\int \frac{\frac{1}{e^{\ln(\cos(t))}}}{\sin^2(t)} \right) \sin(t) \\
&= \left(\int \frac{\frac{1}{\cos(t)}}{\sin^2(t)} \right) \sin(t) \\
&= \left(\int \frac{1}{\cos(t)\sin^2(t)} \right) \sin(t) \\
&= \left(\int \left(\frac{1}{\cos^2(t)} \right) \left(\frac{\cos(t)}{\sin^2(t)} \right) \right) \sin(t) \\
&= \left(\int \left(\frac{1}{1-\sin^2(t)} \right) \left(\frac{\cos(t)}{\sin^2(t)} \right) \right) \sin(t) \\
&= \left(\int \frac{-du}{1-\frac{1}{u^2}} \right) \sin(t) \qquad \text{Use } u = \frac{1}{\sin(x)}, \text{ then } \sin(x) = \frac{1}{u} \\
&= \left(\int \frac{-u^2}{u^2-1} du \right) \sin(t) \qquad \text{(multiply top and bottom by } u^2\text{)} \\
&= \left(\int -1 + \frac{-1}{2(u-1)} + \frac{1}{2(1+u)} \right) \sin(t) \\
&= \left(-u + \frac{1}{2}\ln|u+1| - \frac{1}{2}\ln|u-1| \right) \sin(t) \\
&= \left(\frac{-1}{\sin(t)} + \frac{1}{2}\ln\left| \frac{1}{\sin(t)} + 1 \right| - \frac{1}{2}\ln\left| \frac{1}{\sin(t)} - 1 \right| \right) \sin(t) \\
&= \left(\frac{-1}{\sin(t)} + \frac{1}{2}\ln\left| \frac{\frac{1}{\sin(t)} + 1}{\frac{1}{\sin(t)} - 1} \right| \right) \sin(t) \\
&= \left(\frac{1}{-\sin(t)} + \frac{1}{2}\ln\left| \frac{1+\sin(t)}{1-\sin(t)} \right| \right) \sin(t) \qquad \text{(multiply top and bottom by } \sin(t)\text{)} \\
&= -1 + \sin(t)\coth^{-1}(\sin(t))
\end{aligned}
$$

(d) Notice that the equation $y'' - \tan(t)y' + 2y = 0$, although quite complicated, is still *linear*. What is the general solution of $y'' - \tan(t)y' + 2y = 0$?

$$y(t) = Ay_1(t) + By_2(t) = A\sin(t) + B(-1 + \sin(t)\coth^{-1}(\sin(t)))$$