MATH 54 - MOCK FINAL EXAM

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Name: _____

Instructions: This is a mock final, designed to give you extra practice for the actual final. Good luck!!!

1		20
2		20
3		20
4		10
5		15
6		20
7		10
8		10
Bonus		5
Total		125

1. *(20 = 15 + 5 points)*
   (a) Find a diagonal matrix D and an orthogonal matrix P such that $A = PDP^T$, where:

   $$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

   (b) Use (a) to write the quadratic form $3x_1^2 + 6x_2^2 + 3x_3^2 - 4x_1x_2 + 8x_1x_3 + 4x_2x_3$ without cross-product terms.

2. *(20 points, 2 points each)*

   Mark the following statements as **TRUE** or **FALSE**. If the statement is **TRUE**, don't do anything. If the statement is **FALSE**, provide an explicit counterexample.

   (a) If A is a 3×3 matrix with eigenvalues $\lambda = 0, 2, 3$, then A must be diagonalizable!

   (b) There does not exist a 3×3 matrix A with eigenvalues $\lambda = 1, -1, -1 + i$.

   (c) If A is a symmetric matrix, then all its eigenvectors are orthogonal.

   (d) If Q is an orthogonal $n \times n$ matrix, then $Row(Q) = Col(Q)$.

   (e) The equation $A\mathbf{x} = \mathbf{b}$, where A is a $n \times n$ matrix always has a unique least-squares solution.

(f) If $AB = I$, then $BA = I$.

(g) If A is a square matrix, then $Rank(A) = Rank(A^2)$

(h) If W is a subspace, and $P\mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto W, then $P^2\mathbf{y} = P\mathbf{y}$

(i) If $T : V \to W$, where $dim(V) = 3$ and $dim(W) = 2$, then T cannot be one-to-one.

(j) If A is similar to B, then $det(A) = det(B)$.

3. *(20 points)* Solve the following system $\mathbf{x}' = A\mathbf{x}$, where:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

4. *(10 points)* Solve the following system $\mathbf{x}' = A\mathbf{x}$, where:

$$A = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$$

5. *(15 points)* Assume you're given a coupled mass/spring system with
   $N = 3$, $m_1 = m_2 = m_3 = 1$ and $k_1 = k_2 = k_3 = k_4 = 1$. Find the
   proper frequencies and proper modes.

6. *(20 points)* Solve the following heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & 0 < x < 1, \quad t > 0 \\ u(0,t) = u(1,t) = 0 & t > 0 \\ u(x,0) = x & 0 < x < 1 \end{cases}$$

7. *(10 points)* Solve the following wave equation:

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} = 16\dfrac{\partial^2 u}{\partial x^2} & -\infty < x < \infty, \quad t > 0 \\[2mm]
u(x,0) = e^{-x^2} & -\infty < x < \infty \\[2mm]
\dfrac{\partial u}{\partial t}(x,0) = \sin(x) & -\infty < x < \infty
\end{cases}
$$

Hint: Careful! Do not use separation of variables for this, because $-\infty < x < \infty$. Use d'Alembert's formula!

8. *(10 points)* Solve the following equation using either undetermined coefficients or variation of parameters:

$$y'' + 2y' + y = e^t$$

Bonus *(5 points)*

Disclaimer: This problem is slightly harder than the other ones. It's just meant for the people who're bored and want an extra challenge! Only attempt it if you truly understand vector spaces and linear transformations!

Let V be the vector space of infinitely differentiable functions f from $\mathbb{R}$ to $\mathbb{R}$.

Define $T : V \rightarrow V$ by: $T(y) = y'' - 3y' + 2y$.

(a) Show T is a linear transformation.

(b) Find $Nul(T)$.

(c) Is T one-to-one?

(d) Show T is onto. Namely, given f in V, show that $T(y) = f$
     has at least one solution.

     Hint: Find a formula for y.

(e) Why does this not contradict the theorem in linear algebra that
     says "If T is an onto linear transformation, then T is also one-
     to-one"?