FINAL EXAM (VOJTA) - ANSWER KEY

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(1) I got:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

But many, many other choices are possible, depending on how you row-reduce your matrix!

(2) Let $\mathbf{v_1}$ etc. be the 4 vectors that are given to you.

Suppose $a\mathbf{v_1} + b\mathbf{v_2} = c\mathbf{v_3} + d\mathbf{v_4}$, then $a\mathbf{v_1} + b\mathbf{v_2} - c - \mathbf{v_3} - d\mathbf{v_4} = \mathbf{0}$, hence you'd have to solve:

$$\begin{bmatrix} 1 & 0 & 4 & 2 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 1 & 2 \\ 4 & -1 & -6 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

where $\mathbf{x} = \begin{bmatrix} a \\ b \\ -c \\ -d \end{bmatrix}$.

Solving this equation gives you:

$$\begin{bmatrix} a \\ b \\ -c \\ -d \end{bmatrix} = t \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

where t is any number you'd like (except for 0).

So if you set $t = 3$ for example, you get:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \\ -3 \end{bmatrix}$$

Hence your vector is:

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$$a\mathbf{v_1} + b\mathbf{v_2} = \begin{bmatrix} -2 \\ 0 \\ -5 \\ -9 \end{bmatrix}$$

Check:

$$c\mathbf{v_3} + d\mathbf{v_4} = \begin{bmatrix} -2 \\ 0 \\ -5 \\ -9 \end{bmatrix}$$

(3) $ad - bc \neq 0$.

First of all, by IMT, all we need to show is that $\{a\mathbf{u} + b\mathbf{v}, c\mathbf{u} + d\mathbf{v}\}$ is linearly independent.

So suppose $\lambda_1(a\mathbf{u} + b\mathbf{v}) + \lambda_2(c\mathbf{u} + d\mathbf{v}) = \mathbf{0}$.

Then $(\lambda_1 a + \lambda_2 c)\mathbf{u} + (\lambda_1 b + \lambda_2 d)\mathbf{v} = \mathbf{0}$

But because $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent, we get:

$$\begin{cases} \lambda_1 a + \lambda_2 c = 0 \\ \lambda_1 b + \lambda_2 d = 0 \end{cases}$$

That is: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

But now if $ad - bc \neq 0$, this matrix is invertible, and we get $\lambda_1 = \lambda_2 = 0$, hence linear independence, and if $ad - bc = 0$, then we get linear dependence!

(4)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{3} \\ -2 \\ 1 \end{bmatrix}$$

The reason for this is that the matrix A only has rank 2 (and does not have rank 3), hence $A^T A$ only has rank 2.

(5) -4

(6)

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, Q = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

(7) $\cos(\theta) = \frac{\mathbf{x}\cdot\mathbf{y}}{\|x\|\|y\|} = \frac{1}{\sqrt{14}}$, so $\boxed{\theta = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right)}$, where $\mathbf{x} = (1,2,3), \mathbf{y} = (1,0,0)$.

(8) $(0, \frac{\pi}{2})$ (remember to divide your equation by $\tan(t)$)

(9) $A = PDP^{-1}$, where:

$$D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, P = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

Now form:

$$\mathbf{X}(t) = \begin{bmatrix} e^{2t} & e^{-t} \\ e^{2t} & -2e^{-t} \end{bmatrix}$$

And the matrix Φ we're looking for is:

$$\Phi(t) = \mathbf{X}(t)\mathbf{X}^{-1}(0) = \begin{bmatrix} e^{2t} & e^{-t} \\ e^{2t} & -2e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} e^{2t} & e^{-t} \\ e^{2t} & -2e^{-t} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}e^{2t} + \frac{1}{3}e^{-t} & \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} \\ \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t} & \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} \end{bmatrix}$$

(10) The eigenvalues of A are $-1 \pm 2i$, hence the solutions spiral (because of the imaginary part $2i$), but because because the real part -1 is negative, the solutions will eventually go to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Note: This is basically because your solutions are of the form $e^{-t}\cos(2t)\mathbf{a} + e^{-t}\sin(2t)\mathbf{b}$.

(11)

$$\mathbf{x}(t) = Ae^{-t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + Be^{2t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + Ce^{2t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

(12)

$$A_0 = \frac{1}{2}$$

$$A_m = \frac{2}{\pi m}\sin\left(\frac{\pi m}{2}\right) = \begin{cases} 0 & \text{if } m \text{ is even} \\ (-1)^k & \text{if } m = 2k-1 \text{ is odd} \end{cases}$$

That is:

$$f(x) \text{``}=\text{''} \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2(-1)^k}{\pi(2k-1)} \cos\left(\pi(2k-1)x\right)$$

(13) (a)

$$\begin{cases} X''(x) = \lambda X(x) \\ 2T'(t) - T''(t) = \lambda T(t) \end{cases}$$

(b) $\lambda = -(\pi m)^2$

$X(x) = \sin(\pi m x)$

$T(t) = A_m e^{1+\sqrt{1+(\pi m)^2}} + B_m e^{1-\sqrt{1+(\pi m)^2}}$

Hence:

$$u(x,t) = \sum_{m=1}^{\infty} \left(A_m e^{1+\sqrt{1+(\pi m)^2}} + B_m e^{1-\sqrt{1+(\pi m)^2}} \right) \sin(\pi m x)$$